This chapter discusses equilibrium and yield conditions for membrane elements.

In the first part, as a repetition of Stahlbeton I, the equilibrium conditions are established and the yield conditions for orthogonally reinforced membrane elements are derived.

Subsequently, yield conditions for inclined reinforcement are shown.

Finally, the yield conditions for orthogonally reinforced membrane elements - which assume a constant concrete compressive strength - are extended to account for the influence of the strain state on the concrete compressive strength.

As in the lecture Stahlbeton I, membrane elements are considered in the plane \((x, z)\), since this corresponds to the situation of the girder of a web (longitudinal axis of the girder in \(x\)-direction). Therefore, stresses \(\{\sigma_x, \sigma_y, \tau_{xy}\}\) or membrane forces \(\{n_x, n_z, n_{xz}\} = h\cdot\{\sigma_x, \sigma_z, \tau_{xz}\}\) are investigated (\(h =\) membrane element thickness). Of course, the equilibrium and transformation formulas can also be formulated analogously for membrane elements in the plane \((x, y)\) (stresses \(\{\sigma_x, \sigma_y, \tau_{xy}\}\) and membrane forces \(\{n_x, n_y, n_{xy}\} = h\cdot\{\sigma_x, \sigma_y, \tau_{xy}\}\)).
Membrane elements - Equilibrium

Equilibrium conditions

Equilibrium in directions $x, z$:

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + q_x = 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} + q_z = 0
\]

Or in membrane forces
($\sigma, \tau$ constant over membrane element thickness $h$):

\[
\frac{\partial n_x}{\partial x} + \frac{\partial n_{xz}}{\partial z} + h \cdot q_x = 0 \\
\frac{\partial n_{xz}}{\partial x} + \frac{\partial n_z}{\partial z} + h \cdot q_z = 0 \\
(n_x = h \sigma_x, \quad n_z = h \sigma_z, \quad n_{xz} = h \tau_{xz})
\]

With (moment condition $M_z = 0$):

\[
\tau_{xz} = \tau_{zx} \quad \text{resp.} \quad n_{xz} = n_{zx}
\]

Repetition Stahlbeton I:

- Equilibrium conditions for membrane elements

- Formulation in stresses \( \{ \sigma \} \) or in membrane forces \( \{ n \} \) with \( \{ n \} = h \cdot \{ \sigma \} \)

(with the membrane element thickness $h$ (often also defined as $t$ or $b_w$))
Repetition Stahlbeton I:

- Stress transformation and representation in the Mohr’s Circle

- Principal directions and principal stresses (directions with $\tau_{in} = 0$, maximum / minimum values of normal stress)

- The sign convention in Mohr’s circle differs from the usual convention
Membrane elements - Stress transformation

Stress transformation: Mohr’s circle

\[
\sigma_x = \frac{\sigma_x + \sigma_z}{2} + \frac{\sigma_x - \sigma_z}{2} \cos 2\phi + \tau_{xy} \sin 2\phi
\]

\[
\sigma_y = \frac{\sigma_x + \sigma_z}{2} + \frac{\sigma_y - \sigma_z}{2} \cos 2\phi - \tau_{xy} \sin 2\phi
\]

\[
\tau_{m} = -\frac{\sigma_x - \sigma_z}{2} \sin 2\phi + \tau_{xy} \cos 2\phi
\]

Cos 2\(\phi\) = cos^2 \(\phi\) - sin^2 \(\phi\)

1 = sin^2 \(\phi\) + cos^2 \(\phi\)

sin 2\(\phi\) = 2 sin \(\phi\) cos \(\phi\)
Membrane elements - Stress transformation

Stress transformation: Mohr’s circle

\[ \sigma_x = \frac{\sigma_z + \sigma_y}{2} + \frac{\sigma_z - \sigma_y}{2} \cos 2\phi + \tau_{xy} \sin 2\phi \]
\[ \sigma_y = \frac{\sigma_z + \sigma_x}{2} + \frac{\sigma_z - \sigma_x}{2} \cos 2\phi - \tau_{xy} \sin 2\phi \]
\[ \tau_{xy} = -\frac{\sigma_z - \sigma_x}{2} \sin 2\phi + \tau_{xy} \cos 2\phi \]

Centre / Radius Mohr’s circle

\[ \tau_{xy} = \tau_{xy} = 0 \rightarrow \phi_1 = \frac{1}{2} \tan^{-1} \left( \frac{2\tau_{xy}}{\sigma_y - \sigma_x} \right) \]
\[ \sigma_{ij} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left( \frac{\sigma_z - \sigma_x}{2} \right)^2 + 4\tau_{xy}^2} \]

Centre / Radius Mohr’s circle

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Membrane elements - Equilibrium

**Equilibrium (reinforced concrete = concrete + reinforcement)**
Orthogonally reinforced element (reinforcement directions x, z):

- Concrete is homogeneous and isotropic, absorbs compressive stresses ≤ $f_c$ in any direction but no tensile stresses
- Reinforcement only carries forces in the direction of the bar, up to a maximum value $f_s$ and is distributed and anchored in such a way that equivalent distributed stresses can be expected
- Perfect bond between concrete and reinforcement

In membrane forces:

\[
\begin{align*}
  n_x &= n_{y,x} + n_{z,x} = n_{y,x} + a_x \sigma_{y,x} \\
  n_y &= n_{x,y} + n_{z,y} = n_{x,y} + a_y \sigma_{x,y} \\
  n_z &= n_{x,z} + n_{y,z} = n_{x,z} + a_z \sigma_{x,z} \\
  n_x &= h \sigma_{x} \quad n_y = h \sigma_{y} \quad n_z = h \tau_{x,y}
\end{align*}
\]

In equivalent stresses:

\[
\begin{align*}
  \sigma_x &= \sigma_{x,x} + \rho_x \sigma_{y,x} \\
  \sigma_y &= \sigma_{y,y} + \rho_y \sigma_{x,y} \\
  \tau_{x,y} &= \tau_{x,y} + \rho_{x,y} \sigma_{x,y}
\end{align*}
\]

(reinforcement ratios $\rho_x = a_{x,1}/h$, $\rho_z = a_{z,1}/h$)

Repetition Stahlbeton I: Forces in orthogonally reinforced membrane elements

- The applied load must correspond to the sum of the forces in concrete and reinforcement.

- Coordinate axes in reinforced concrete are conventionally selected such that they coincide with the reinforcement directions (usually x-axis in the direction of the stronger reinforcement). With inclined reinforcement, the x-axis coincides with one reinforcement direction.

- Orthogonal reinforcement in direction of coordinate axes x and z does not contribute to $n_{xz}$, i.e. $n_{xz} = 0$.

- Instead of forces, the formulation can be expressed in equivalent stresses (concrete stresses and stresses in the reinforcement multiplied by the respective geometrical reinforcement ratio, which corresponds to the membrane forces divided by the element thickness).

Additional remark:

- When strictly deriving equivalent stresses, a correction term should also be introduced for concrete stresses (analogous to the factor $[1-\rho]$ for normal force), since not the entire membrane element thickness is available. For normal membrane forces in the x- and z-directions, this would be $(1- \rho_x)$ and $(1- \rho_z)$. Since an inclined stress field in relation to the reinforcement directions usually arises (concrete compression can be transferred as transverse compression via reinforcement), this correction term is usually neglected. However, it can be seen that the concrete stress field is disturbed by the reinforcement.
Membrane elements - Equilibrium

Equilibrium («reinforced concrete = concrete + reinforcement»)
Orthogonally reinforced element (reinforcement directions x, z):

Representation with Mohr’s circles (straightforward for orthogonal reinforcement, since $\tau_{xz} = 0$):

\[
\begin{align*}
\sigma_x &= \sigma_{x\nu} + \rho_x \sigma_{x\nu} = \sigma_{c3} \cos^2 \alpha + \rho_x \sigma_{x\nu} \\
\sigma_z &= \sigma_{z\nu} + \rho_z \sigma_{z\nu} = \sigma_{c3} \sin^2 \alpha + \rho_z \sigma_{z\nu} \\
\tau_{xz} &= \tau_{x\nu z} = -\sigma_{c3} \sin \alpha \cos \alpha
\end{align*}
\]

$\alpha$: Principal direction of concrete compression

Repetition Stahlbeton I: Forces in orthogonally reinforced membrane elements

- Behaviour is not isotropic, not even with "isotropic reinforcement" (same reinforcement in both directions)!

- "Shear" is related to the direction of the (x-) reinforcement!
Yield conditions for isotropic materials (e.g. steel: Tresca, von Mises) cannot be applied to reinforced concrete, neither for the dimensioning of reinforcement and not even if it is "isotropic" (equal reinforcement ratio in both directions). Their application can be on the safe or unsafe side.

There is an obvious difference, for example, for the behaviour under combined tensile/compressive actions, where loading in one reinforcement direction does not influence the resistance of the reinforcement in the other reinforcement direction; according to Tresca / von Mises, on the other hand, compressive resistance in one direction is reduced by tensile loading acting perpendicularly to it (and vice versa). On the other hand, a material with yield conditions according to Tresca / von Mises has a shear resistance, which orthogonal (isotropic) reinforcement does not have.
Repetition Stahlbeton I (or Baustatik): Bending and normal force

Interaction diagrams of reinforced concrete beams under bending and normal force can be determined for perfectly plastic behaviour by means of a graphical linear combination of the non-plastic domains of concrete and reinforcement.
Rectangular cross-section - rigid-perfectly plastic behaviour, without concrete cover, \( A_s = A_s' \)

(2) Reinforcement

→ Non-plastic domain \( Y_s < 0 \) for two reinforcement layers is a parallelogram (for symmetrical reinforcement \( A_s = A_s' \) → rhombus), which is defined by the vectors corresponding to the two reinforcement layers.

→ Graphical combination of the two reinforcement layers by geometric linear combination (see combination of concrete and reinforcement)

→ Corner points: both reinforcements yield, sides: one reinforcement yields

→ Plastic strain increments are orthogonal to the yield surface \( Y_s = 0 \), directed outwards (to the yield surface), i.e. gradients

Repetition Stahlbeton I (or Baustatik): Bending and normal force
Rectangular cross-section - rigid-perfectly plastic behaviour, without concrete cover, $A_x = A'_x$

(3) Reinforced concrete = concrete + reinforcement

→ Yield surface of reinforced concrete obtained by geometric linear combination of the yield surfaces $Y_c = 0$ and $Y_s = 0$
→ Procedure: Move the yield surface ($Y_c = 0$) with its origin along yield surface ($Y_s = 0$)
(or vice versa $Y_s = 0$ along $Y_c = 0$)
→ Resulting area $Y < 0$ corresponds to the non-plastic domain of the reinforced concrete cross-section, it is at least weakly convex, the associated flow rule (orthogonality of the plastic strain increments with respect to yield surface) still applies
→ One reinforcement remains elastic (rigid) along the straight segments of the yield surface.
→ Procedure transferable to any component and stresses

Repetition Stahlbeton I (or Baustatik): Bending and normal force
Repetition Stahlbeton I: Yield conditions of orthogonally reinforced membrane elements:

- Yield conditions of orthogonally reinforced shear elements can be determined analogously to the interaction diagrams for bending and normal force by a graphical linear combination of the non-plastic domains of concrete and reinforcement.

- Non-plastic domain of orthogonal reinforcement: rectangle in the reinforcement plane \( n_{xz} = \tau_{xz} = 0 \)

- Non-plastic domain of the concrete: two elliptical cones (front \( \sigma_{c1} = 0 \), back \( \sigma_{c3} = f_c \) )
Membrane elements - Yield conditions

Yield condition for orthogonally reinforced membrane elements
Geometric linear combination concrete + reinforcement

\[ n_x = n_{xx} + n_{xz} \]
\[ n_z = n_{zx} + n_{zz} \]
\[ n_{xz} = n_{xx} \sigma_{xz} \]

Procedure:
Move the yield surface \( Y_c = 0 \) with its origin along yield surface \( Y_s = 0 \) (or vice versa \( Y_s = 0 \) along \( Y_c = 0 \))

Repetition Stahlbeton I: Yield conditions of orthogonally reinforced membrane elements
Membrane elements - Yield conditions

Yield conditions / yield Regimes reinforced concrete

Linear combination of the yield conditions, i.e. shifting the yield condition of the concrete (origin) along the yield condition of the reinforcement. «reinforced concrete = steel + concrete»

\[
\begin{align*}
Y_1 &= n_{xz}^2 - (a_{xz} f_{xz} - n_y)(a_{xz} f_{xz} - n_y) = 0 \\
Y_2 &= n_{xz}^2 - (h f_c - a_{xz} f_{xz} + n_y)(a_{xz} f_{xz} - n_y) = 0 \\
Y_3 &= n_{xz}^2 - (a_{xz} f_{xz} - n_y)(h f_c - a_{xz} f_{xz} + n_y) = 0 \\
Y_4 &= n_{xz}^2 - (h f_c / 2)^2 = 0 \\
Y_5 &= n_{xz}^2 + (a_{xz} f_{xz} + n_y)(h f_c + a_{xz} f_{xz} + n_y) = 0 \\
Y_6 &= n_{xz}^2 + (h f_c + a_{xz} f_{xz} + n_y)(a_{xz} f_{xz} + n_y) = 0 \\
Y_7 &= n_{xz}^2 - (h f_c + a_{xz} f_{xz} + n_y)(h f_c + a_{xz} f_{xz} + n_y) = 0
\end{align*}
\]

SN: Reinforcement areas per unit length in x- and z-direction \( a_{xz} = A_{xz} / s_z \) \( a_{xz} = A_{xz} / s_x \)

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Repetition Stahlbeton I: Yield conditions of orthogonally reinforced membrane elements
Membrane elements - Yield conditions

Yield conditions / yield Regime Reinforced concrete

Y1: Both reinforcements yield in tension
\( \sigma_{sx} = f_{sx}, \ \sigma_{sz} = f_{sz}, \ 0 \geq \sigma_{cz} \geq -f_c \)

Y2: z-reinforcement yields in tension, concrete crushes
\( \sigma_{sz} = f_{sz}, \ \sigma_{cz} = -f_c, -f_{sx} \leq \sigma_{sx} \leq f_{sx} \)

Y3: x-reinforcement yields in tension, concrete crushes
\( \sigma_{sx} = f_{sx}, \ \sigma_{cz} = -f_c, -f_{sz} \leq \sigma_{sz} \leq f_{sz} \)

Y4: Concrete crushes
\( \sigma_{cz} = -f_c, -f_{sx} \leq \sigma_{sx} \leq f_{sx}, -f_{sz} \leq \sigma_{sz} \leq f_{sz} \)

Y5: x-reinforcement yields in compression, concrete crushes
\( \sigma_{sx} = -f_{sx}, \ \sigma_{cz} = -f_c, -f_{sz} \leq \sigma_{sz} \leq f_{sz} \)

Y6: z-reinforcement yields in compression, concrete crushes
\( \sigma_{sz} = -f_{sz}, \ \sigma_{cz} = -f_c, -f_{sx} \leq \sigma_{sx} \leq f_{sx} \)

Y7: Both reinforcements yield in compression, concrete crushes
\( \sigma_{sx} = -f_{sx}, \ \sigma_{sz} = -f_{sz}, \ \sigma_{cz} = -f_c \)

(mean concrete principal stress also negative)

SN: failure type: very ductile / ductile (except for very flat stress field inclinations) / brittle

Repetition Stahlbeton I: Yield conditions of orthogonally reinforced membrane elements
Membrane elements - Yield conditions

Strain increments and principal compressive direction

Strain increments are proportional to the components of the outer normal to the yield surface (gradient) in the respective point of the yield surface ($\kappa \geq 0$: any factor):

$$\dot{\varepsilon}_x = \kappa \frac{\partial Y}{\partial n_x}, \quad \dot{\varepsilon}_z = \kappa \frac{\partial Y}{\partial n_z}, \quad \dot{\gamma}_w = \kappa \frac{\partial Y}{\partial n_w}$$

Inclination $\alpha$ of the principal compressive direction $3$ with respect to the $x$-axis follows from the Mohr’s circle of plastic strain increments (principal strain direction = principal compressive direction in concrete):

$$\cot 2\alpha = \frac{\dot{\varepsilon}_x - \dot{\varepsilon}_z}{\dot{\gamma}_w} \quad \text{mit} \quad \cot \alpha = \frac{\cos(2\alpha) + 1}{\sin(2\alpha)} = \cot(2\alpha) + \frac{\cos^2(2\alpha) + \sin^2(2\alpha)}{\sin^2(2\alpha)}$$

Repetition Stahlbeton I: Yield conditions of orthogonally reinforced membrane elements
Membrane elements - Yield conditions

Dimensioning of reinforcement

Design practice: Usually Regime 1 (ductile failure type; both reinforcements yield before concrete crushing, concrete remains "elastic" = undamaged).

Yield condition for Regime 1 in parametric form (→ direct dimensioning):

\[ Y_i = n_i^2 - (a_i f_{\text{cc}} - n_i) (a_i f_{\text{cc}} - n_i) = 0 \]

\[ k = \cot \alpha \quad a_i f_{\text{cc}} \geq n_i + k |n_i| \]

Yield condition in Regime 1 is governing (no concrete crushing) if:

\[ h f_c \geq a_i f_{\text{cc}} + a_i f_{\text{cc}} - (n_i + n_i) \]

SN:

→ Value of \( f_c \) see next slides. Approximation according to SIA 262: \( f_c = k_c f_{\text{cd}} \) (with \( k_c = 0.55 \))

→ Inclination of the concrete stress field in Regime 1 follows:

\[ \cot \alpha = \frac{(a_i f_{\text{cc}} - n_i) (a_i f_{\text{cc}} - n_i)}{n_i} \]

→ Value \( k = \cot \alpha \) can theoretically be freely chosen, in design standards often limited by the condition \( 0.5 \leq k \leq 2 \)

→ Use of \( k = 1 \), i.e. \( \alpha = 45^\circ \), "linearised yield conditions", implemented in many FE programs. Safe dimensioning, but this is just one of many possibilities (possibly strongly on the safe side)
Membrane elements - Yield conditions

Web crushing (Regime 2)

If the condition $hf_z \geq a_{zw} f_{zw} + a_{zz} f_{zz} - (n_z + n_z)$ is not satisfied, a failure type where the concrete fails under compression (crushes) is governing.

Regime 2 is also of particular practical relevance. It applies if the condition $a_{zw} f_{zw} - n_z > a_{zz} f_{zz} - n_z$ is met.

→ Type of failure: Yielding of the $z$-reinforcement with simultaneous concrete compressive failure, called web crushing.

→ The corresponding limitation of the shear resistance of the membrane element can be represented as a quarter circle.

→ Limitations for $\cot \alpha$ correspond to straight lines in the diagram.

SN: Figure on the right = projection of the yield surface to the plane $(n_z, n_{zz})$, shifted by $a_{zw} f_{zw}$ ($n_z$ = generalised reaction)

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Repetition Stahlbeton I: Design of orthogonally reinforced membrane elements in Regime 2
Membrane elements - Yield conditions

Skew reinforcements

With skew reinforcements, the determination of the yield conditions becomes mathematically significantly more complicated. For Regime 1, for example, the yield condition becomes:

\[ Y_i = \left( \tau_{\psi} - \rho_{s} f_{s} \sin \psi \cos \psi \right)^2 - \left( \rho_{s} f_{s} + \rho_{s} f_{w} \cos^2 \psi - \sigma_{s} \right) \left( \rho_{s} f_{s} \sin^2 \psi - \sigma_{s} \right) = 0 \]

With loads transformed to skew coordinates, the relationships for the direct design of the reinforcement in Regime 1 follow from:

\[ k = \left| \cos \psi + \sin \psi \cos \theta \right| \]

\[ \rho_{s} f_{s} \geq \frac{1}{\sin \psi} \left( \sigma_{s} + k \sigma_{\psi} \right) \]

and for checking the concrete compressive strength:

\[ \sigma_{c3} = \frac{1}{\sin \psi} \left[ 2 \tau_{\psi} \cos \psi - \sigma_{\psi} (k + k^{-1}) \right] \geq -f_{c} \]

This and the following two slides show how skew reinforcements can be designed and how the corresponding yield conditions can be determined [Seelhofer (2009)]. By the term “skew reinforcement”, we refer here to oblique reinforcement directions (as opposed to orthogonal reinforcement), and NOT to orthogonal reinforcement simply rotated with respect to the coordinate axes, as sometimes called “skew reinforcement” in literature.

The relationships are more complicated than for orthogonal reinforcement, since the reinforcement inclined with respect to the \((x,y)\)-axes bears part of the shear loads \(n_{xz}\) (unlike orthogonal reinforcement, for which \(n_{xz} = 0\)).

If the stress is transformed using skew coordinates (in the direction of reinforcement), the design can be performed in the same way as for orthogonal reinforcement (direct dimensioning). For the sake of simplicity, the coordinates axes are selected so that one reinforcement direction runs in the \(x\)-direction.

The given relationships were derived from Seelhoffer and Marti and are much more practical than older "design algorithms" for skew reinforcements, as they are implemented in FE programs today (in the case they allow a design of skew reinforcements at all).
Similarly as for orthogonal reinforcement, the yield conditions of membrane elements reinforced in a number of arbitrary directions can also be determined by a graphical linear combination of the non-plastic zones of concrete and reinforcement.

The non-plastic domain of each reinforcement layer is a straight line. While these straight lines are parallel to the coordinate axes for reinforcements in the coordinate directions (here: $x$-reinforcement), they are inclined for skew reinforcement. They are neither in the plane, $n_{xz} = 0$, nor are their projections into this plane parallel to one of the coordinate axes. The corner points of a reinforcement inclined by the angle $\psi_k$ with respect to the $x$-axis are defined by the membrane forces \( \{n_x, n_y, n_{xz}\}_{sk} = \{a_{sk}f_{sk}\cos^2\psi_k, a_{sk}f_{sk}\sin^2\psi_k, a_{sk}f_{sk}\sin\psi_k\cos\psi_k\} \) corresponding to its tensile yield force $a_{sk}f_{sk}$ or by the equivalent stresses \( \{\rho_kf_{sk}\cos^2\psi_k, \rho_kf_{sk}\sin^2\psi_k, \rho_kf_{sk}\sin\psi_k\cos\psi_k\} \) (see bottom left figure).

These two straight lines of the reinforcements in direction $x$ and $k$ can be combined to a parallelogram which lies in the plane defined by the two straight lines (see top right figure).

With three inclined reinforcement layers, the non-plastic domain of the reinforcement corresponds to a parallelepiped (see bottom right figure). For even more reinforcement directions, the yield surface can be constructed as a graphical linear combination of the yield figures of the individual reinforcement layers.

(continued on the following slide)
Membrane elements - Yield conditions

Skew reinforcements

→ Two oblique = skew reinforcement layers: flat, parallelogram-shaped yield figure of the reinforcement, inclined relative to the $\sigma_{x-x}$-$\sigma_{y-y}$-plane.

→ Three oblique = skew reinforcement layers: Parallelepiped, can carry load without concrete (imagine a hinged connection of reinforcing bars)

→ Yield surface of a reinforced concrete membrane element results from translation of the concrete yield surface with its origin on the yield surface of the reinforcement (geometric linear combination)

Alternatively, $n$ inclined reinforcements can be transformed to the orthogonal $x-z$ coordinate system to determine an equivalent orthogonal reinforcement. The yield conditions of the orthogonal reinforcement can then be used, whereby the applied stress is to be transformed into the direction of the equivalent orthogonal reinforcement.

(see dissertation Seelhofer, 2009)

As with orthogonal reinforcement, the non-plastic domain of the concrete consists of two elliptical cones (front $\sigma_{c1} = 0$, rear $\sigma_{c3} = -f_c$). Combining the non-plastic domains of concrete and reinforcement results in the yield surface of the inclined = skew reinforced membrane element (see figure above).

Alternatively, skew reinforcements can be replaced by an equivalent orthogonal reinforcement. Assuming a constant stress state in the reinforcement (i.e., yielding), the stress state can be transformed to the $x-z$ coordinate system, even for a reinforcement mesh consisting of any number of non-orthogonal reinforcement layers (see equations in the figure).

The equivalent steel stresses of the equivalent orthogonal reinforcement then correspond to the principal stresses ($\sigma_{s1}, \sigma_{s2}$) of the stress state defined by $(\sigma_{sx}, \sigma_{sy}, \tau_{sxy})$. The directions of the equivalent orthogonal reinforcement correspond to the associated principal directions $(1, 2)$.

For the equivalent orthogonal reinforcement, the yield conditions for orthogonal reinforcement can be applied. For the verifications, the loads shall be transformed into the directions $(1, 2)$ of the reinforcement.
On this and the following two slides (from Kaufmann (1998)) the influence of a strain state dependent compressive strength on the resistance of orthogonally reinforced membrane elements is investigated.

Basically, the consideration of a compressive strength dependent on the strain state requires carrying out load-deformation analyses (see chapter "Load-deformation behaviour of membrane elements"). However, the influence can be analytically approximated with some simplifications. Investigations with a refined model (Cracked Membrane Model CMM) confirm this observation.

It is important to note that a compressive strength dependent on the strain state has no influence on the load-bearing resistance in Regime 1, since the resistance in this Regime is uniquely determined by the yielding of the two reinforcements. However, it does influence the limits of Regime 1 and the resistance in Regimes 2, 3, etc.

The figure on the left shows the yield condition for orthogonally reinforced membrane elements with constant compressive strength. The area in which failure occurs due to yielding of the two reinforcements (Regime 1) is triangular (bounded by the yellow line). With a compressive strength dependent on the strain state (lower compressive strength at large transverse tensile strains), early concrete failure occurs with very flat (or steep) stress field inclinations. Since the transverse strains are larger with such inclinations, the area of Regime 1 is therefore more narrow in this case.

The figure in the middle shows calculations with the CMM (tedious, each point of the diagram corresponds to a full nonlinear load-deformation analysis). The figure on the right shows the approximation presented on the next page. This fits well with the more precise calculations.
Membrane elements - Yield conditions

Concrete compressive strength

The yield surface can be modified taking into account the dependence of the concrete compressive strength on the transverse strains.

→ Area of Regime 1 is reduced (affected: zones with very flat / steep inclinations)

→ Calculation with Cracked Membrane Model (CMM, middle row) is tedious

→ Approximate solution (bottom row):

assuming: \( k_c f_c = \frac{(f_c')^{1/3}}{0.4 + 30 \cdot \epsilon_i} \) (1998):

\[
Y_1: \quad \tau_{1}^{H} = (\rho f_y - \sigma_n) (\rho f_y - \sigma_n) \\
Y_2: \quad \tau_{1}^{L} = (\rho f_y - \sigma_n) \left\{ \frac{2.0 + 25}{3 (\rho f_y - \sigma_n)} \right\}^{2} - \frac{29}{12} \\
Y_3: \quad \tau_{2}^{H} = (\rho f_y - \sigma_n) \left\{ \frac{2.0 + 25}{3 (\rho f_y - \sigma_n)} \right\}^{1/2} - \frac{29}{12} \\
Y_4: \quad \tau_{2}^{L} = \left\{ \frac{25}{29} (f_c')^{1/3} \right\}^{1/2}
\]

On the right side, the yield conditions for ideally plastic behaviour and constant concrete compressive strength is shown on top. The failure conditions considering the deformation-dependency of the concrete compressive strength is shown below it (same figures as on previous slide).

The approximation (bottom figures) is based on the assumption that at the Regime boundary the concrete compression is \( \epsilon_3 = -\epsilon_{cu} = -0.002 \) and the stronger reinforcement just reaches the yield point, taking into account the tensile stiffness with a factor of 0.8 \( (\epsilon_{sm} = 0.8 \cdot f_s / E_s) \).

Thus the principal strain follows at the Regime boundary 1-2 with \( \epsilon_3 = -\epsilon_{cu} \) and \( \epsilon_x = \epsilon_{sm} \) from the relationship \( \epsilon_1 = \epsilon_x + (\epsilon_x - \epsilon_3) \cot^2 \alpha \), since the compression field inclination is known. At the Regime boundary both reinforcements just yield when the concrete crushes.

By inserting into the equation for the effective concrete compressive strength, the equations presented in the slide are obtained. For the Regime limit, these equations agree very well (since the assumptions made apply to the strains at the Regime limit). For areas further away, the approximation is still sufficient.
Membrane elements - Yield conditions

Concrete compressive strength

The yield surface can be modified taking into account the dependence of the concrete compressive strength on the transverse strains.

→ Area of Regime 1 is reduced (affected: zones with very flat / steep inclinations)

→ Calculation with Cracked Membrane Model (CMM, middle row) is tedious

→ Approximate solution (bottom row):

according to SIA: (2013), \( k_c f_c = \frac{f_c}{1.2 + 55 \cdot \varepsilon_i} \):

\[
\begin{aligned}
Y_1 & : \quad \tau_{w1} = (\rho_c f_{ck} - \sigma) \\frac{f_c}{11 (\rho_c f_{ck} - \sigma)} \quad \text{(unchanged)} \\
Y_2 & : \quad \tau_{w2} = (\rho_c f_{ck} - \sigma) \frac{f_c}{22 (\rho_c f_{ck} - \sigma)} \quad \text{[modified]} \\
Y_3 & : \quad \tau_{w3} = (\rho_c f_{ck} - \sigma) \frac{f_c}{22 (\rho_c f_{ck} - \sigma)} \quad \text{[modified]} \\
Y_4 & : \quad \tau_{w4} = 0.17 \frac{f_c}{16 (\rho_c f_{ck} - \sigma)} \quad \text{[d.h. } \tau_{w4} = 0.327 \cdot f_c = \frac{0.65 \cdot f_c}{2} \text{]} \\
\end{aligned}
\]

Limit Regime 1: concrete crushes, stronger reinforcement at yield surface  \( \tau_c \) determinable

The equations shown on the previous slide apply to the relationship given there for the effective concrete compressive strength.

The relationship for the effective concrete compressive strength according to the standard SIA 262 results in the equations given above. These can also be used for a fully code-compliant design in Regimes 2, 3 and 4.
Membrane elements - Yield conditions

Summary

→ «Reinforced concrete = steel + concrete»
→ Well suited for design on the basis of FE calculations (limit values)
→ Dimensioning for Regime 1, verification of the prerequisites (no concrete crushing, compressive strength, see stress fields or load-deformation behaviour)
→ Safe design with linearised yield conditions possible
→ Regime 2 important for beams, "web crushing failure".
→ Skew reinforcements can be treated in the same way (but mathematically more complicated)