5 Slabs

In-depth study and additions to Stahlbeton II

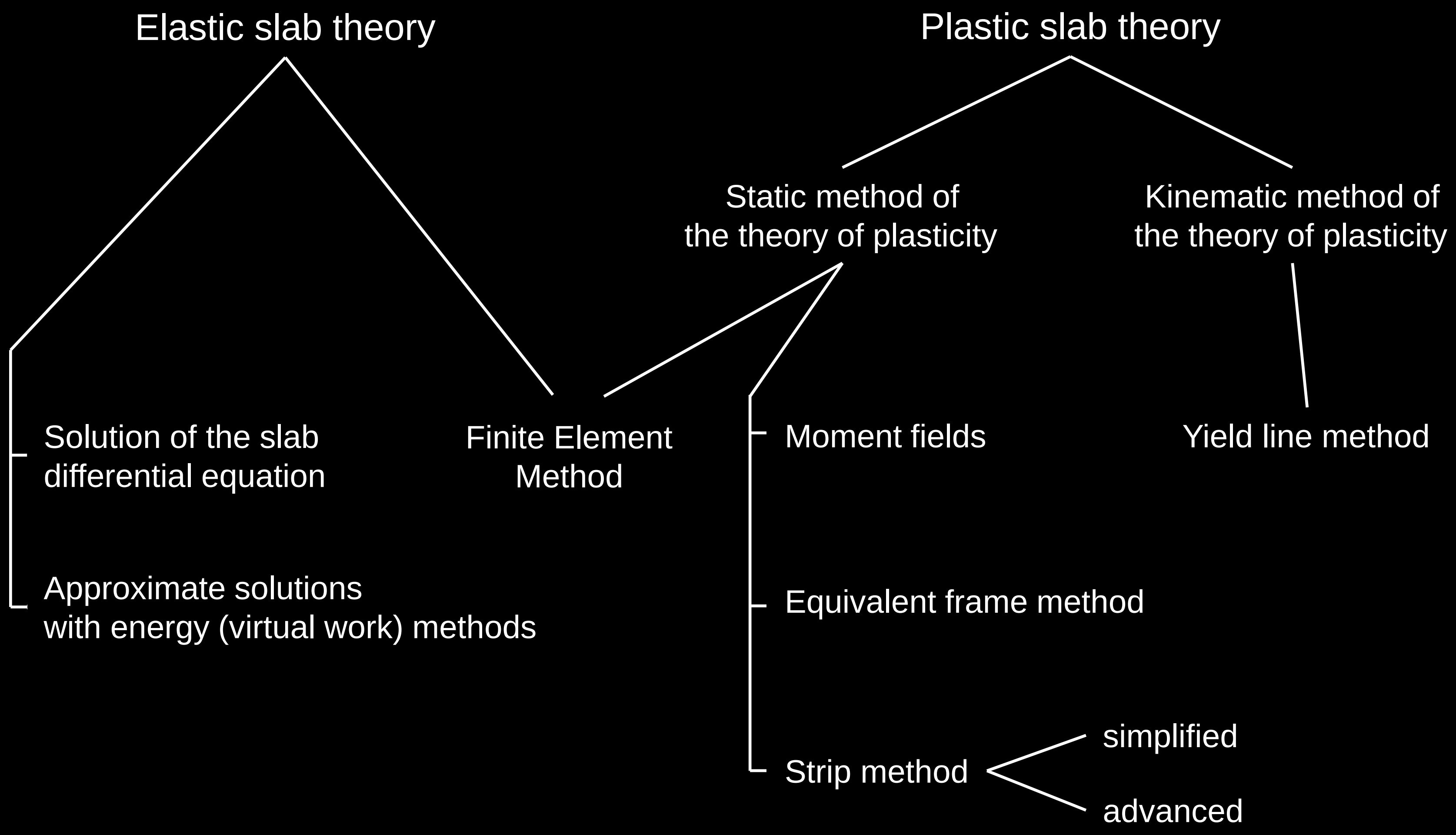
Learning objectives

Within this chapter, the students are able to:

- design and assess slabs with orthogonal and skew reinforcement based on elastic and plastic slab theory and thereby
  - elaborate on the applicability, accuracy, and limitations of the used approaches.
  - explain the underlying differences of the methods, especially with respect to the treatment of twisting moments.
- illustrate in terms of Mohr's circles the superposition of the bending resistance of two layers of orthogonal or skew reinforcement and explain how it results in the bending moment yield conditions.
- identify the necessity and how to design edge reinforcement in slab corners and edges.

Slabs - Basics

Structural analysis / Calculation methods - Overview



Elastic slab theory

Plastic slab theory

Static method of
the theory of plasticity

Kinematic method of
the theory of plasticity

Solution of the slab
differential equation

Finite Element
Method

Moment fields

Yield line method

Approximate solutions
with energy (virtual work) methods

Equivalent frame method

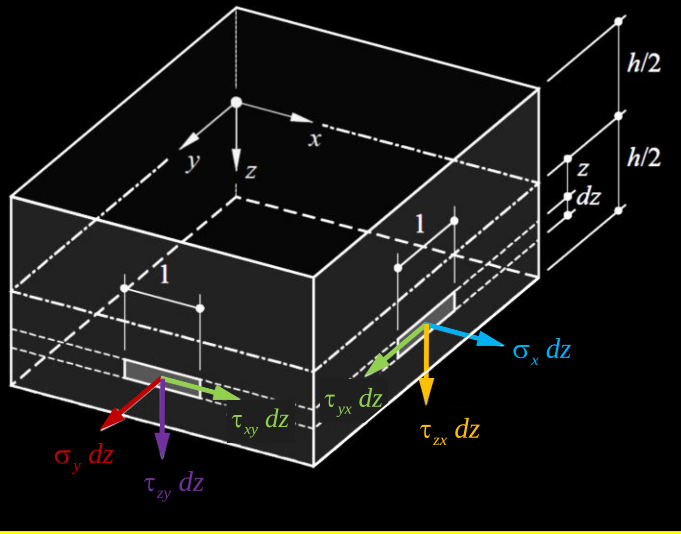
Strip method

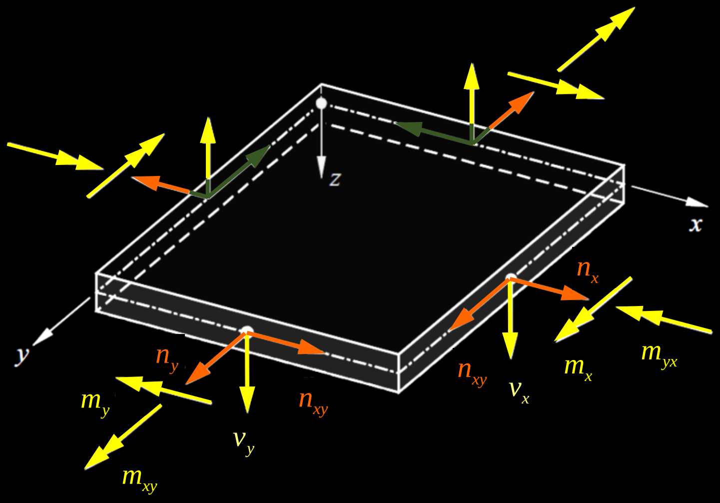
simplified

advanced

Slabs - Basics

Plane elements - Stress resultants





$$m_x = \int_{-h/2}^{h/2} \sigma_x z\,dz\,, \quad m_y = \int_{-h/2}^{h/2} \sigma_y z\,dz\,, \quad m_{xy} = m_{yx} = \int_{-h/2}^{h/2} \tau_{xy} z\,dz \quad [\text{kNm/m} = \text{kN}]$$

$$v_x = \int_{-h/2}^{h/2} \tau_{zx}\,dz\,, \quad v_y = \int_{-h/2}^{h/2} \tau_{zy}\,dz \quad [\text{kN/m}]$$

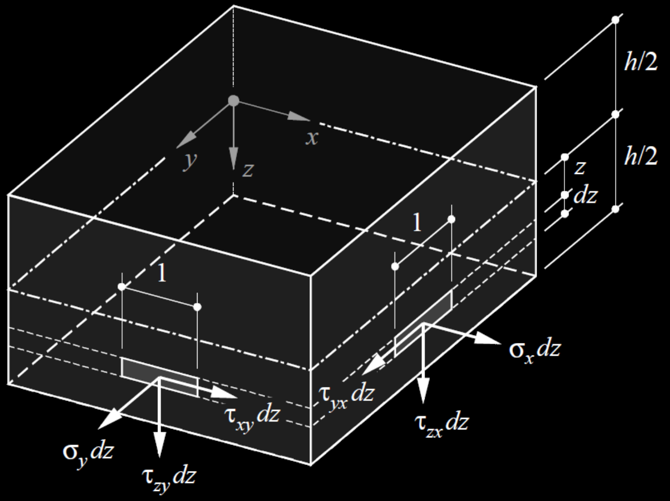
Bending stress state (slab): bending moments and shear forces

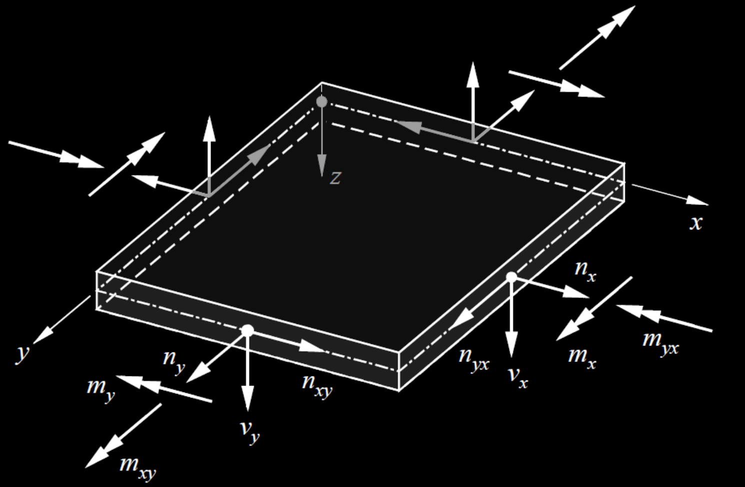
$$n_x = \int_{-h/2}^{h/2} \sigma_x\,dz\,, \quad n_y = \int_{-h/2}^{h/2} \sigma_y\,dz\,, \quad n_{xy} = n_{yx} = \int_{-h/2}^{h/2} \tau_{xy}\,dz \quad [\text{kN/m}]$$

Membrane stress state (membrane element): membrane forces (normal/shear forces)

Slabs - Basics

Plane elements - Stress resultants





$$m_x = \int_{-h/2}^{h/2} \sigma_x z\, dz, \quad m_y = \int_{-h/2}^{h/2} \sigma_y z\, dz, \quad m_{xy} = m_{yx} = \int_{-h/2}^{h/2} \tau_{xy} z\, dz$$

$$v_x = \int_{-h/2}^{h/2} \tau_{zx}\, dz, \quad v_y = \int_{-h/2}^{h/2} \tau_{zy}\, dz$$

$$n_x = \int_{-h/2}^{h/2} \sigma_x\, dz, \quad n_y = \int_{-h/2}^{h/2} \sigma_y\, dz, \quad n_{xy} = n_{yx} = \int_{-h/2}^{h/2} \tau_{xy}\, dz$$

Sign convention

- Positive stresses act on elements with positive outer normal direction in positive axis direction
- Positive membrane and shear forces correspond to positive associated stresses
- Positive moments correspond to positive associated stresses for z > 0
- Indices: 1st index: direction of stress
  2nd index: normal direction of the element at which stress is applied

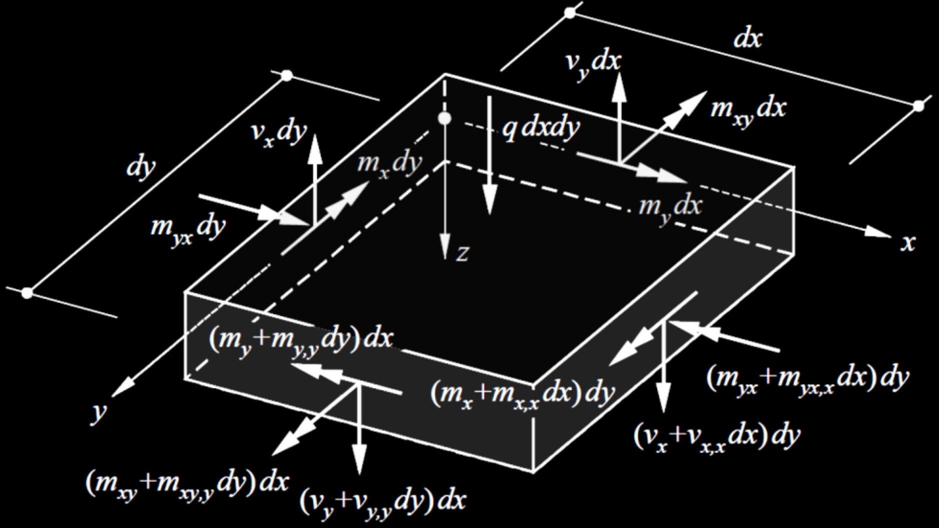
5 Slabs

In-depth study and additions to Stahlbeton II

5.1 Equilibrium conditions

Slabs - Equilibrium

Equilibrium conditions - Cartesian coordinates



Equilibrium condition for slabs:

$$\underbrace{\boxed{\frac{\partial^2 m_x}{\partial x^2}}}_{\text{beam in x-direction}} + 2\underbrace{\frac{\partial^2 m_{xy}}{\partial x \partial y}}_{\text{additionally: twisting moments}} + \underbrace{\frac{\partial^2 m_y}{\partial y^2}}_{\text{beam in y-direction}} + q = 0$$

$$\uparrow$$

Derivation via equilibrium at the differential slab element:

$$-v_x\,dy - v_y\,dx + \left(v_y + \frac{\partial v_y}{\partial y}dy\right)dx + \left(v_x + \frac{\partial v_x}{\partial x}dx\right)dy + q\,dxdy = 0 \quad \longrightarrow \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + q = 0$$

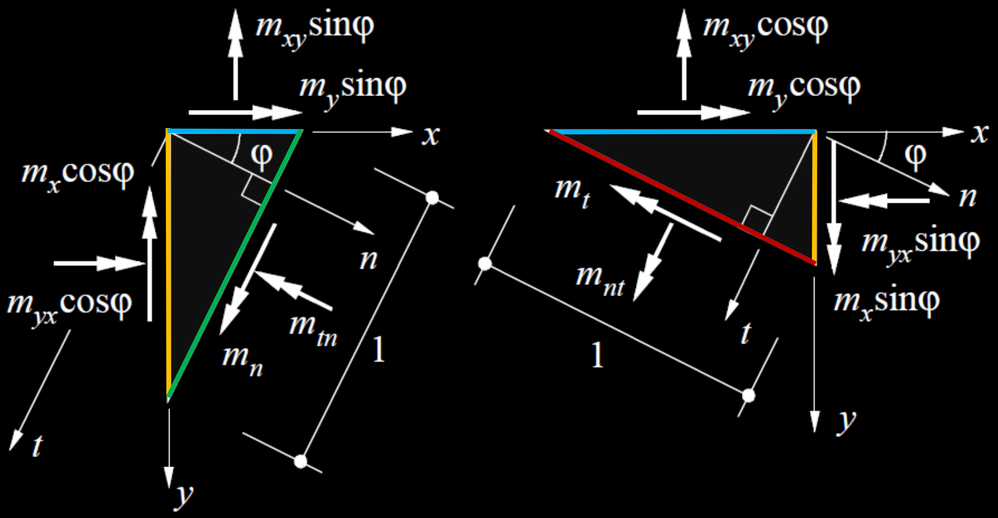
$$-m_x\,dy - m_{xy}\,dx + \left(m_x + \frac{\partial m_x}{\partial x}dx\right)dy + \left(m_{xy} + \frac{\partial m_{xy}}{\partial y}dy\right)dx - v_x dydx = 0 \quad \longrightarrow \quad \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y} - v_x = 0$$

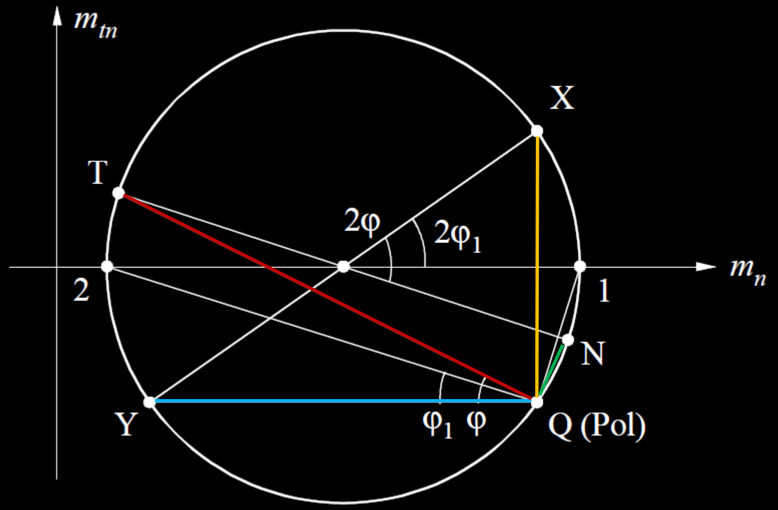
$$-m_y\,dx - m_{yx}\,dy + \left(m_y + \frac{\partial m_y}{\partial y}dy\right)dx + \left(m_{yx} + \frac{\partial m_{yx}}{\partial x}dx\right)dy - v_y dxdy = 0 \quad \longrightarrow \quad \frac{\partial m_y}{\partial y} + \frac{\partial m_{yx}}{\partial x} - v_y = 0$$

terms with *(dx)*^{2} or *(dy)*^{2} neglected

Slabs - Equilibrium

Stress transformation: Bending and twisting moments





Bending and twisting moments in any direction φ:

$$m_n = m_x \cos^2\varphi + m_y \sin^2\varphi + m_{xy}\sin 2\varphi$$

$$m_t = m_x \sin^2\varphi + m_y \cos^2\varphi - m_{xy}\sin 2\varphi$$

$$m_{tn} = \left(m_y - m_x\right)\sin\varphi\cos\varphi + m_{xy}\cos 2\varphi$$

NB: $\sin 2\varphi = 2\sin\varphi\cos\varphi,\ \cos 2\varphi = \cos^2\varphi - \sin^2\varphi$

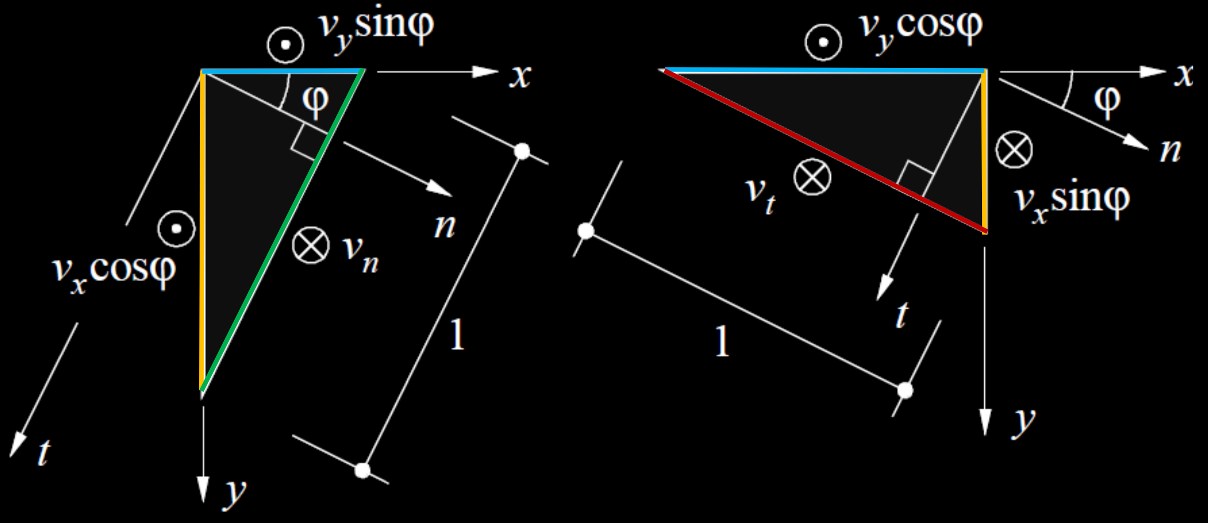
Principal direction φ_1 (twisting moments = 0) and principal moments (Mohr's circle):

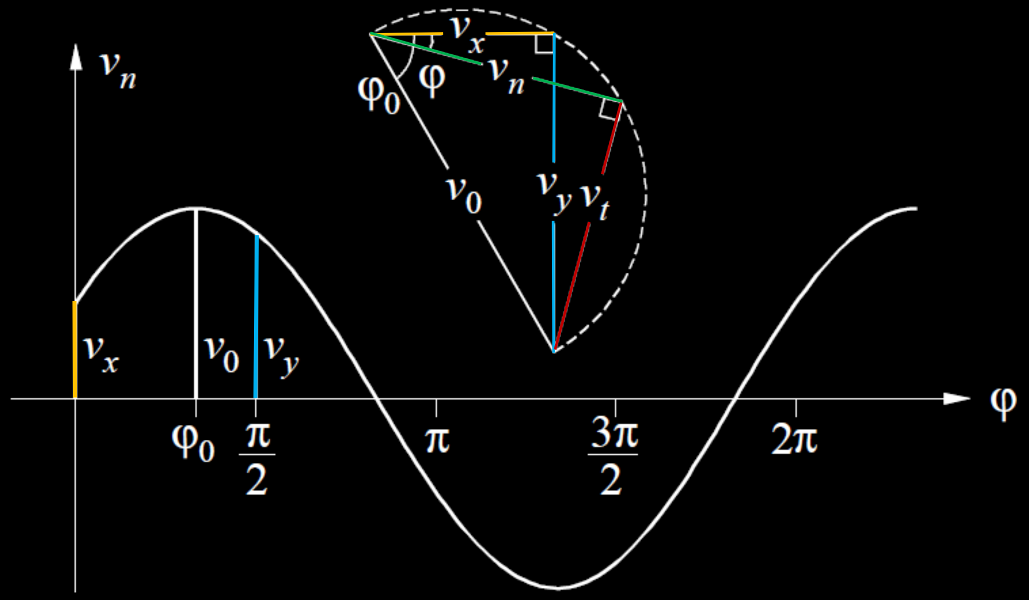
$$\tan 2\varphi_1 = \frac{2m_{xy}}{m_x - m_y}$$

$$m_{1,2} = \frac{m_x + m_y}{2} \pm \frac{\sqrt{\left(m_x - m_y\right)^2 + 4m_{xy}^2}}{2}$$

Slabs - Equilibrium

Stress transformation: Shear forces





Shear forces in any direction φ:

$$v_n = v_x \cos\varphi + v_y \sin\varphi$$

$$v_t = -v_x \sin\varphi + v_y \cos\varphi$$

Principal shear force and associated direction φ_0 (interpretation with Thales' circle):

$$v_0 = \sqrt{v_x^2 + v_y^2}$$

$$\tan\varphi_0 = \frac{v_y}{v_x}$$

(generally $\varphi_0 \neq \varphi_1$)

Slabs - Boundary conditions

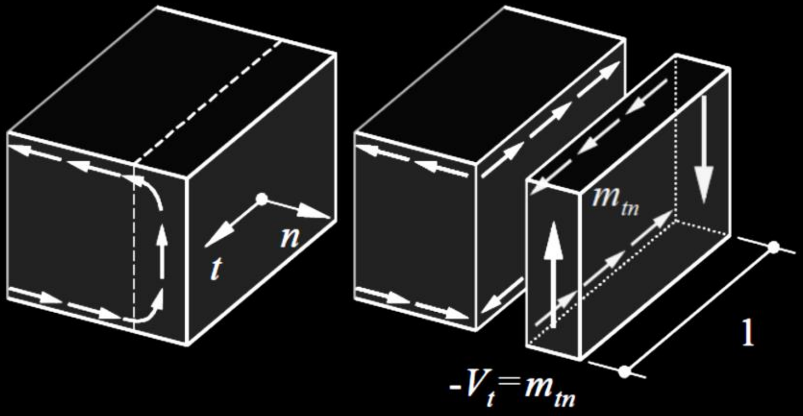
Boundary conditions based on equilibrium

Static method of the theory of plasticity - Explanation of load-bearing effect in the region of slab edges, which is based only on equilibrium considerations:

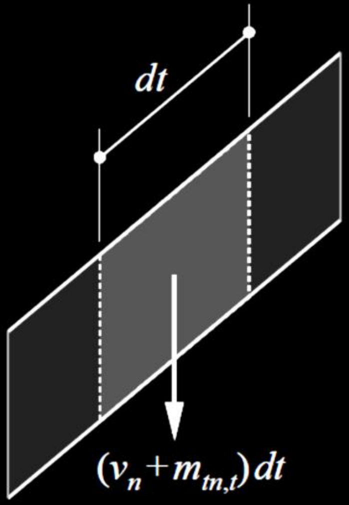
→ From equilibrium in a narrow edge zone of the slab, one gets the edge transverse force: $V_t = -m_{tn}$

→ *If*: The slab edge is stress-free and the stresses σ_t occurring in the edge zone do not change in the *t* direction (Clyde, 1979).

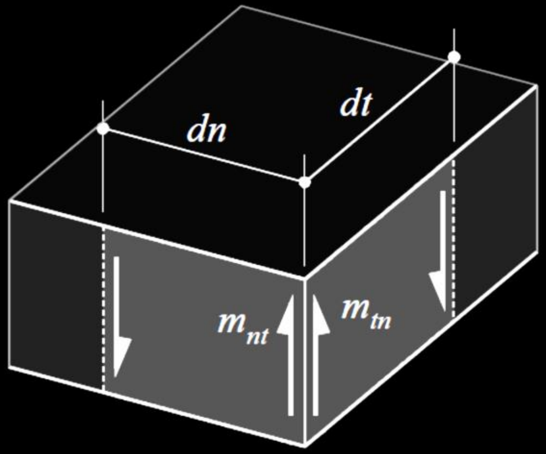
→ From the boundary shear force $V_t = -m_{tn}$, one gets the corner forces 2 m_{tn} and the contribution of $m_{tn,t}$ to the support force.



shear force in boundary zone



support force



corner force

Slabs - Boundary conditions

Boundary conditions on the basis of equilibrium considerations

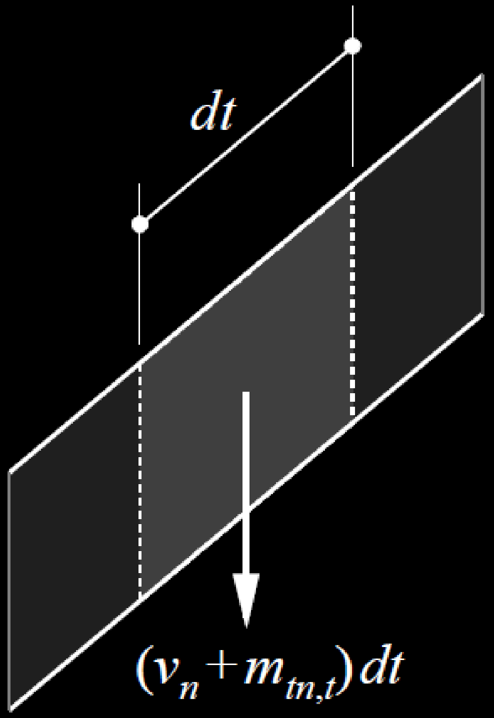
→ Boundary conditions based on equilibrium considerations:

- Clamped edge: m_n, m_{tn} and v_n arbitrary
- Simply supported edge: $m_n = 0$, resulting support force:

$$v_n + \frac{\partial m_{tn}}{\partial t} = \frac{\partial m_n}{\partial n} + 2\frac{\partial m_{nt}}{\partial t}$$

- Free edge: $m_n = 0$, disappearing support force:

$$v_n + \frac{\partial m_{tn}}{\partial t} = \frac{\partial m_n}{\partial n} + 2\frac{\partial m_{nt}}{\partial t} = 0$$



support force

Slabs - Boundary conditions

Edge reinforcement

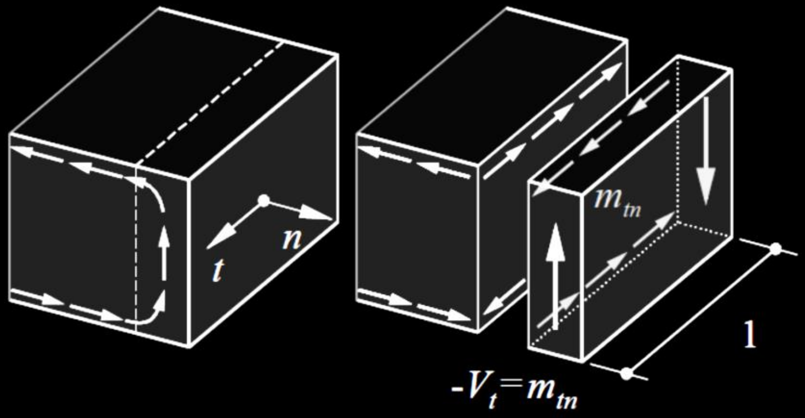
If twisting moments are calculated along simply supported and free edges, a reinforcement must be arranged to carry $V_t = -m_{tn}$.

Figure (corner, pure twisting):

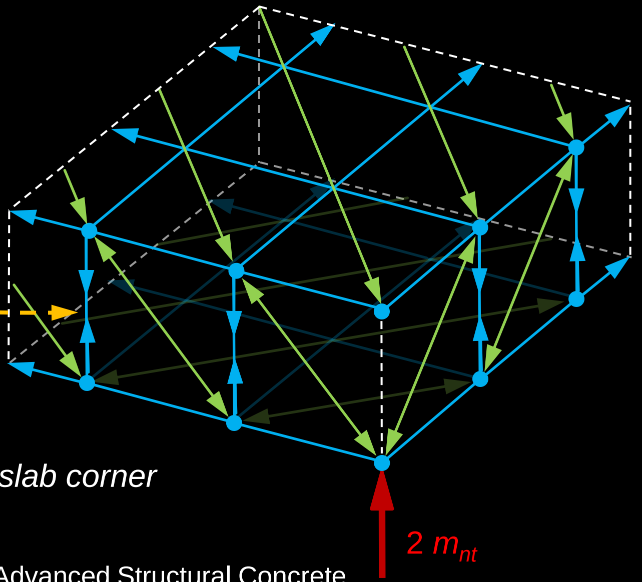
→ Upper and lower side: concrete struts perpendicular to each other, inclined at 45° to the edges of the slabs, support of components normal to the edge by the longitudinal reinforcement.

→ Components in the direction of the slab edges are transferred to the edge members by inclined concrete compression struts. Vertical components correspond to the edge shear forces $V_t = -m_{tn}$

→ Carrying $\underbrace{V_t = -m_{tn}}$ with shear reinforcement or correspondingly detailed bending reinforcement (e.g. «hairpins»).



lateral force in boundary zone



force flow in slab corner

Slabs - Boundary conditions

Discontinuities

Static discontinuity lines are admissible inside the slab (↔ Equivalence of twisting moments at the slab edge and edge shear forces, joining two free slab edges).

At discontinuity lines

→ Bending moments m_n must be continuous ($m_n^+ = m_n^-$)

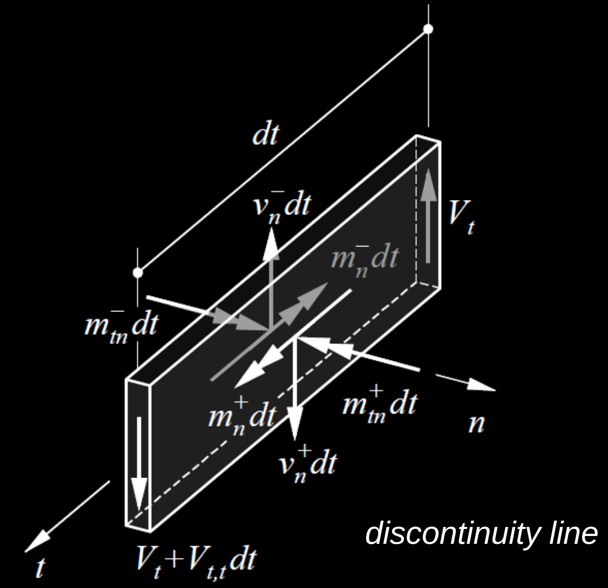
→ Twisting moments m_{nt} and shear forces v_n may be discontinuous (jump) ($m_{nt}^+ \neq m_{nt}^-$, $v_n^+ \neq v_n^-$)

Thus, for a static discontinuity line along which an edge shear force V_t is applied, the following conditions apply:

$$m_n^- = m_n^+$$

$$V_t = m_{nt}^+ - m_{nt}^-$$

$$\frac{\partial V_t}{\partial t} = v_n^- - v_n^+$$



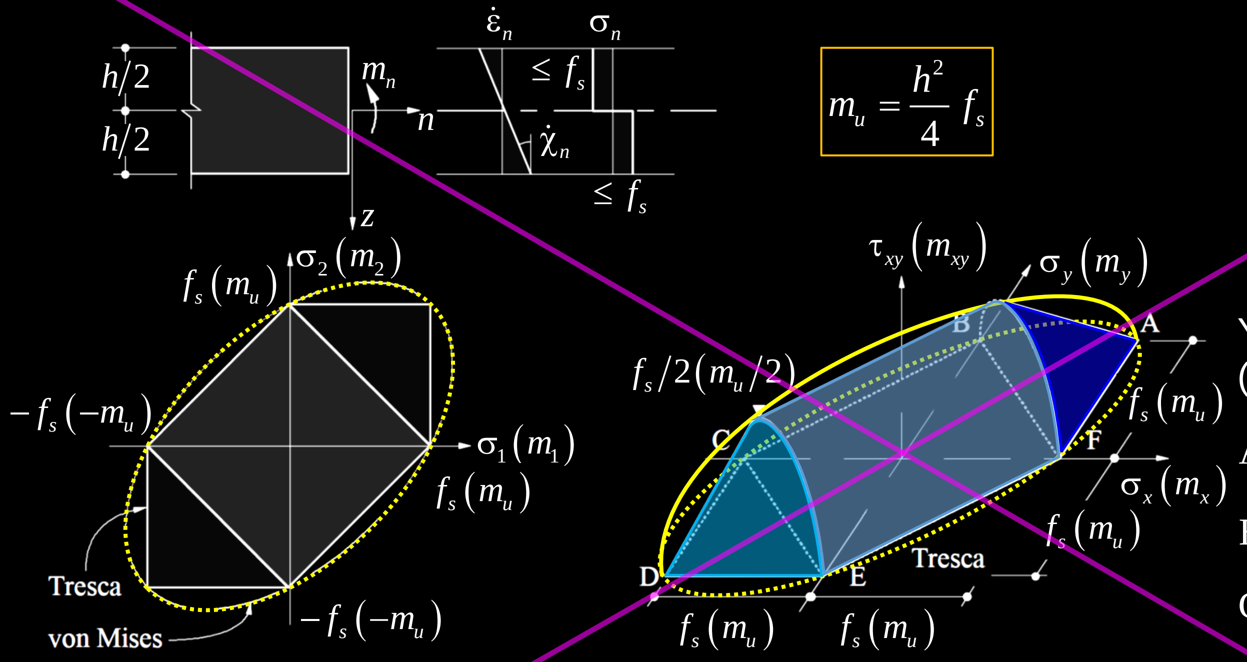
5 Slabs

In-depth study and additions to Stahlbeton II

5.2 Yield conditions

Slabs - Yield conditions

Yield conditions of Tresca and von Mises for isotropic slabs (steel etc.)
(not suitable for reinforced concrete, even with "isotropic reinforcement"!)



$$\boxed{m_u = \frac{h^2}{4} f_s}$$

Yield regimes according to Tresca:
(2 elliptical cones, elliptical cylinder)

$$\text{ABF:} \quad \Phi = (m_u - m_x)(m_u - m_y) - m_{xy}^2 = 0$$

$$\text{BCEF:} \quad \Phi = (m_x - m_y)^2 + 4m_{xy}^2 - m_u^2 = 0$$

$$\text{CDE:} \quad \Phi = (m_u + m_x)(m_u + m_y) - m_{xy}^2 = 0$$

In the fully plasticised state (or rigid-plastic behaviour), the stress state on each side of the median plane is constant → yield condition analogous to the plane stress state:

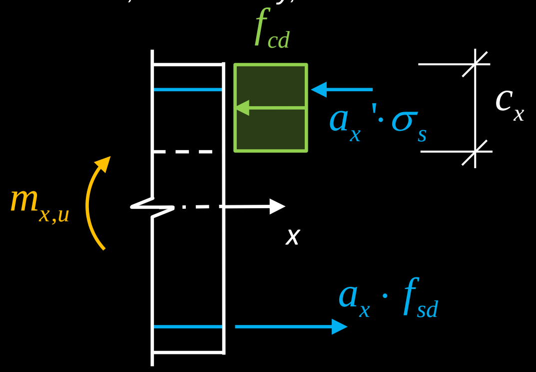
$$\text{T:} \quad \text{Max}(|\sigma_1|,|\sigma_2|,|\sigma_1 - \sigma_2|) - f_s = 0 \quad \rightarrow \quad \text{Max}(|m_1|,|m_2|,|m_1 - m_2|) - m_u = 0$$

$$\text{vM:} \quad \sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2 - f_s^2 = 0 \quad \rightarrow \quad m_x^2 - m_x m_y + m_y^2 + 3m_{xy}^2 - m_u^2 = 0$$

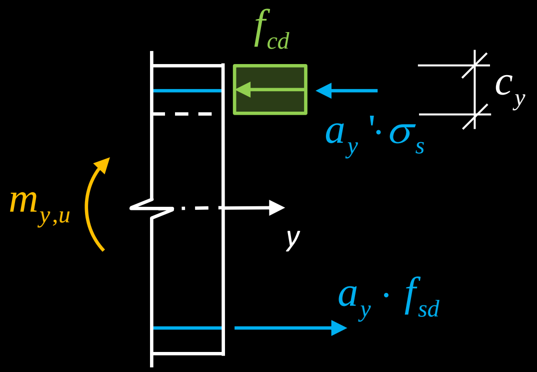
Slabs - Yield conditions

Yield conditions for reinforced concrete slabs

Bending resistances $m_{x,u}$ and $m_{y,u}$ of an orthogonally reinforced slab (reinforcement in *x*- and *y*-direction):



Cross-Section x-direction

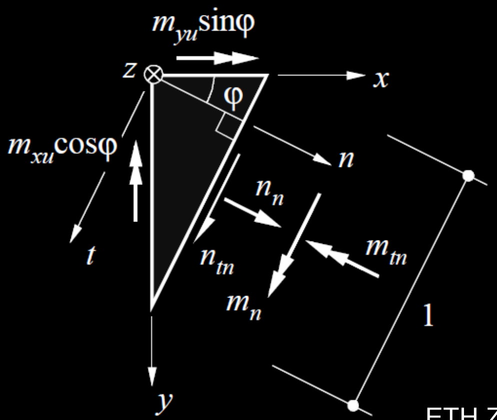


Cross-Section y-direction

Without normal forces, the compression zone heights c_x and c_y and thus $m_{x,u}$ and $m_{y,u}$ result from equilibrium.

Since reinforcement is orthogonal: $m_{xy,u} = 0$

By superposition of the bending resistances in the reinforcement directions and transformation in any direction (analogous to the stress transformations) the bending and twisting moments m_n , m_t and m_{nt} in *n*- and *t*-direction (statically admissible stress state) are obtained:



$$m_n = m_{xu} \cdot \cos^2 \varphi + m_{yu} \cdot \sin^2 \varphi$$

$$m_t = m_{xu} \cdot \sin^2 \varphi + m_{yu} \cdot \cos^2 \varphi$$

$$m_{nt} = \left(m_{yu} - m_{xu}\right) \cdot \sin \varphi \cdot \cos \varphi$$

All membrane forces disappear:

$$n_t = n_n = n_{nt} = 0$$

Slabs - Yield conditions

Yield conditions for reinforced concrete slabs

The resistance is checked on the basis of the normal moments ("normal moment yield condition").

If the compression zone depths are equal, i.e. $c_x = c_y$ the complete solution results:

- Statically admissible stress state (equilibrium)
- Kinematically compatible failure mechanism (yield line, see later)

$$\underbrace{\begin{aligned} m_{n,u} &= m_{x,u} \cdot \cos^2 \varphi + m_{y,u} \cdot \sin^2 \varphi \\ m_{t,u} &= m_{x,u} \cdot \sin^2 \varphi + m_{y,u} \cdot \cos^2 \varphi \end{aligned}}_{\text{Bending resistance for positive bending moments}} \qquad \underbrace{\begin{aligned} m'_{n,u} &= m'_{x,u} \cdot \cos^2 \varphi + m'_{y,u} \cdot \sin^2 \varphi \\ m'_{t,u} &= m'_{x,u} \cdot \sin^2 \varphi + m'_{y,u} \cdot \cos^2 \varphi \end{aligned}}_{\text{Bending resistance for negative bending moments («‘») (the sign of the bending resistance is defined positive)}}$$

For $c_x \neq c_y$ the statically admissible stress state provides a lower limit for the ultimate load:

$$\begin{aligned} m_{n,u} &\geq m_{x,u} \cdot \cos^2 \varphi + m_{y,u} \cdot \sin^2 \varphi \\ m_{t,u} &\geq m_{x,u} \cdot \sin^2 \varphi + m_{y,u} \cdot \cos^2 \varphi \end{aligned} \qquad \begin{aligned} m'_{n,u} &\geq m'_{x,u} \cdot \cos^2 \varphi + m'_{y,u} \cdot \sin^2 \varphi \\ m'_{t,u} &\geq m'_{x,u} \cdot \sin^2 \varphi + m'_{y,u} \cdot \cos^2 \varphi \end{aligned}$$

The differences with regard to the compression zone depths in *x*- and *y*-direction are usually small so that the inequality sign may be suppressed with good approximation.

NB: With a definition range for the angle φ of $\{0 \leq \varphi \leq \pi\}$, the relationship for m_n is sufficient.

Slabs - Yield conditions

Yield conditions for reinforced concrete slabs

The action m_n in the relevant direction φ_u is set equal to the resistance $m_{n,u}$ obtaining:

$$m_{x,u}\cdot\cos^2\varphi_u + m_{y,u}\cdot\sin^2\varphi_u = m_{n,u} \overset{!}{=} m_n = m_x\cdot\cos^2\varphi_u + m_y\cdot\sin^2\varphi_u + 2m_{xy}\cdot\sin\varphi_u\cos\varphi_u$$

Considering that the condition $m_{n,u} \geq m_n$ must be satisfied for all directions φ, the result is (*):

for positive bending moments:

$$\left|\tan\varphi_u\right| = \sqrt{\frac{\left(m_{x,u}-m_x\right)}{\left(m_{y,u}-m_y\right)}}$$

$$\boxed{\begin{aligned} m_{x,u} &= m_x + m_{xy}\cdot\tan\varphi_u \\ m_{y,u} &= m_y + m_{xy}\cdot\cot\varphi_u \end{aligned}}$$

resistance actions

for negative bending moments:

$$\left|\tan\varphi'_u\right| = \sqrt{\frac{\left(m'_{x,u}+m_x\right)}{\left(m'_{y,u}+m_y\right)}}$$

$$\boxed{\begin{aligned} m'_{x,u} &= -m_x - m_{xy}\cdot\tan\varphi'_u \\ m'_{y,u} &= -m_y - m_{xy}\cdot\cot\varphi'_u \end{aligned}}$$

resistance actions

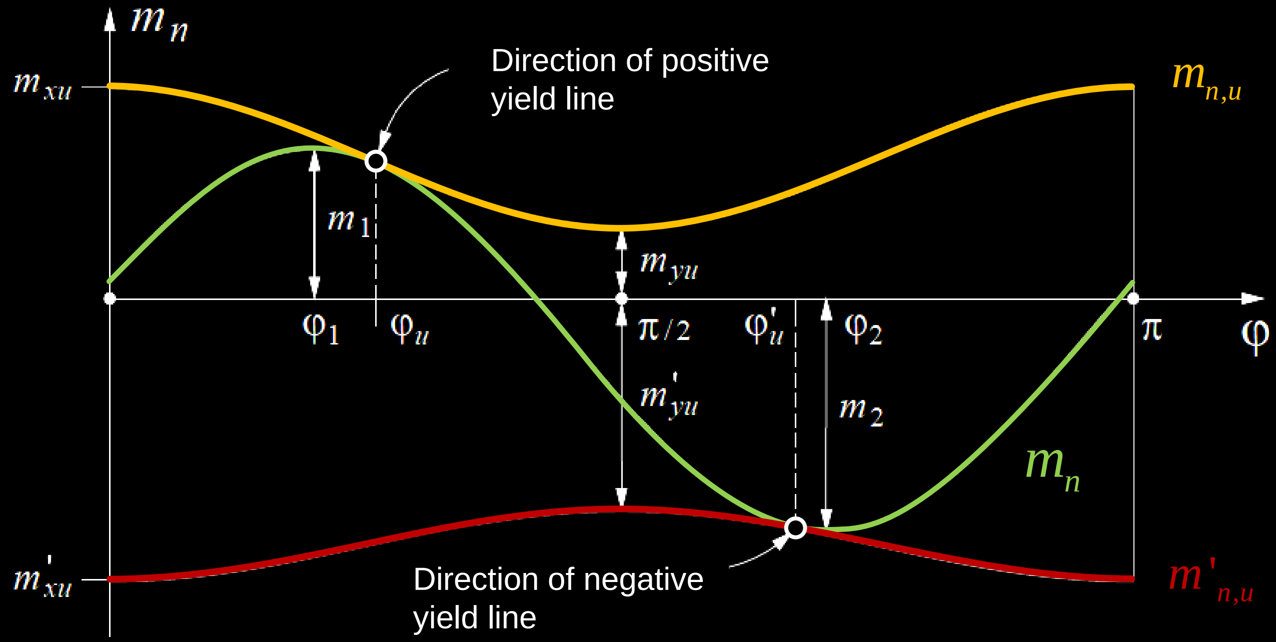
(*) In the relevant direction φ_u (point of contact of $m_{n,u}(\varphi)$ and $m_n(\varphi)$) the difference $m_{n,u} - m_n$ is minimum, thus:

$$m_{n,u}(\varphi) - m_n(\varphi) = \min! \quad \rightarrow \frac{\partial}{\partial\varphi}\left(m_{n,u}(\varphi) - m_n(\varphi)\right) = 0, \quad \frac{\partial}{\partial\varphi}m_{n,u}(\varphi) = \frac{\partial}{\partial\varphi}m_n(\varphi) \quad \rightarrow m_{y,u} - m_{x,u} = m_y - m_x + m_{xy}\left(\cot\varphi_u - \tan\varphi_u\right)$$

after some transformation the specified relations follow by resubstitution.

Slabs - Yield conditions

Yield conditions for reinforced concrete slabs



Bending moments m_n as a function of $\varphi \rightarrow$ Controlling direction φ_u

φ_1, φ_2 → Directions in which the acting positive or negative moment becomes maximum (principal directions for m_n)
φ_u, φ'_u → Directions in which the action curve touches the resistance curve, i.e. $m_n = m_{n,u}$

Generally $\varphi_1 \neq \varphi_u$ resp. $\varphi_2 \neq \varphi'_u$ → Dimensioning of $m_{n,u}$ based on principal moment m_1 is not conservative!

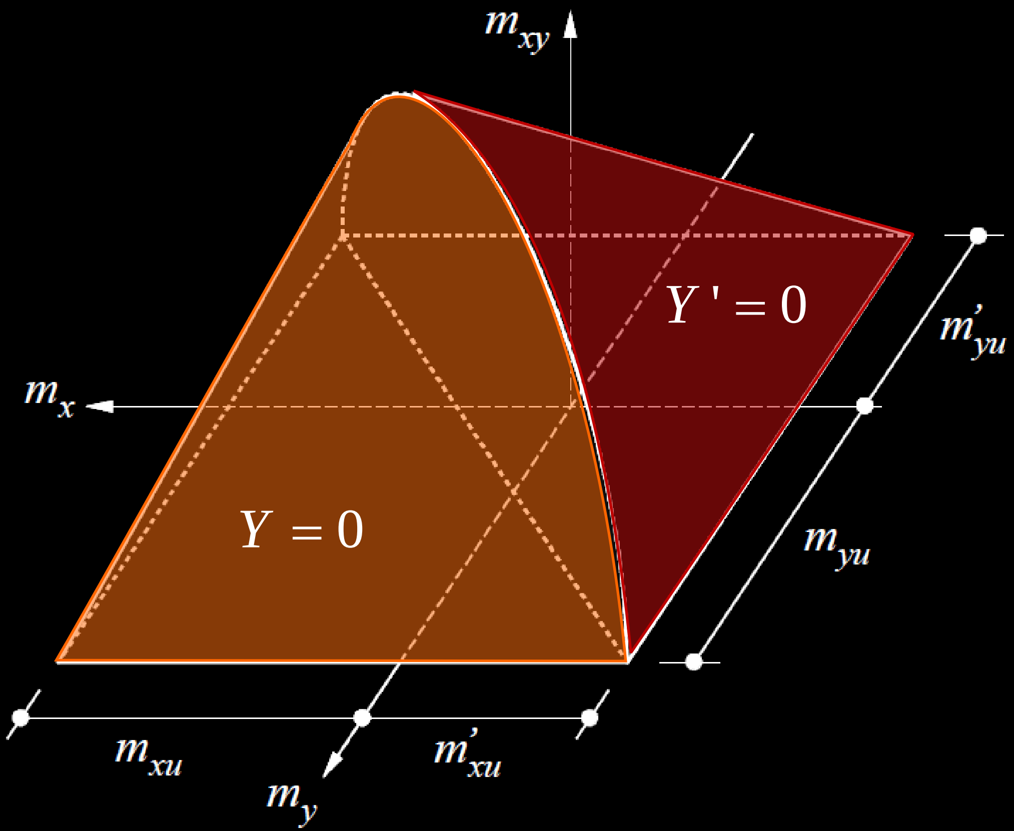
Slabs - Yield conditions

Normal moment yield criterion

If φ_u and φ'_u are eliminated from the previous equations, the normal moment yield criterion results: $-m'_{n,u} \le m_n \le m_{n,u}$

$$\boxed{\begin{aligned} Y &= m_{xy}^2 - \overbrace{\left(m_{x,u} - m_x\right)}^{\ge 0}\overbrace{\left(m_{y,u} - m_y\right)}^{\ge 0} = 0 \\ Y' &= m_{xy}^2 - \underbrace{\left(m'_{x,u} + m_x\right)}_{\ge 0}\underbrace{\left(m'_{y,u} + m_y\right)}_{\ge 0} = 0 \end{aligned}}$$

If $Y \le 0$ and $Y' \le 0$, the yield condition is fulfilled.



The normal moment yield condition forms two elliptical cones in (m_x, m_y, m_{xy}) space. On the conical surfaces $\chi_x \chi_y = 0$ (from yield law), i.e. one of the two principal curvatures disappears. The compatible mechanisms therefore correspond to developable surfaces.

Slabs - Yield conditions

Normal moment yield criterion

If φ_u and φ'_u are eliminated from the previous equations, the normal moment yield criterion results: $-m'_{n,u} \leq m_n \leq m_{n,u}$

$$\boxed{\begin{aligned} Y &= m_{xy}^2 - \overbrace{\left(m_{x,u} - m_x\right)}^{\geq 0}\overbrace{\left(m_{y,u} - m_y\right)}^{\geq 0} = 0 \\ Y' &= m_{xy}^2 - \underbrace{\left(m'_{x,u} + m_x\right)}_{\geq 0}\underbrace{\left(m'_{y,u} + m_y\right)}_{\geq 0} = 0 \end{aligned}}$$

Dito, with notations according to SIA 262:

$$\boxed{\begin{aligned} Y &= m_{xy,d}^2 - \overbrace{\left(m_{x,Rd} - m_{x,d}\right)}^{\geq 0}\overbrace{\left(m_{y,Rd} - m_{y,d}\right)}^{\geq 0} = 0 \\ Y' &= m_{xy,d}^2 - \underbrace{\left(m'_{x,Rd} + m_{x,d}\right)}_{\geq 0}\underbrace{\left(m'_{y,Rd} + m_{y,d}\right)}_{\geq 0} = 0 \end{aligned}}$$

Slabs - Yield conditions

Design moments

The normal moment yield criterion in parametric form: with $k = \left|\tan \varphi_u\right|$ and with $k' = \left|\tan \varphi'_u\right|$

The resulting design moments:

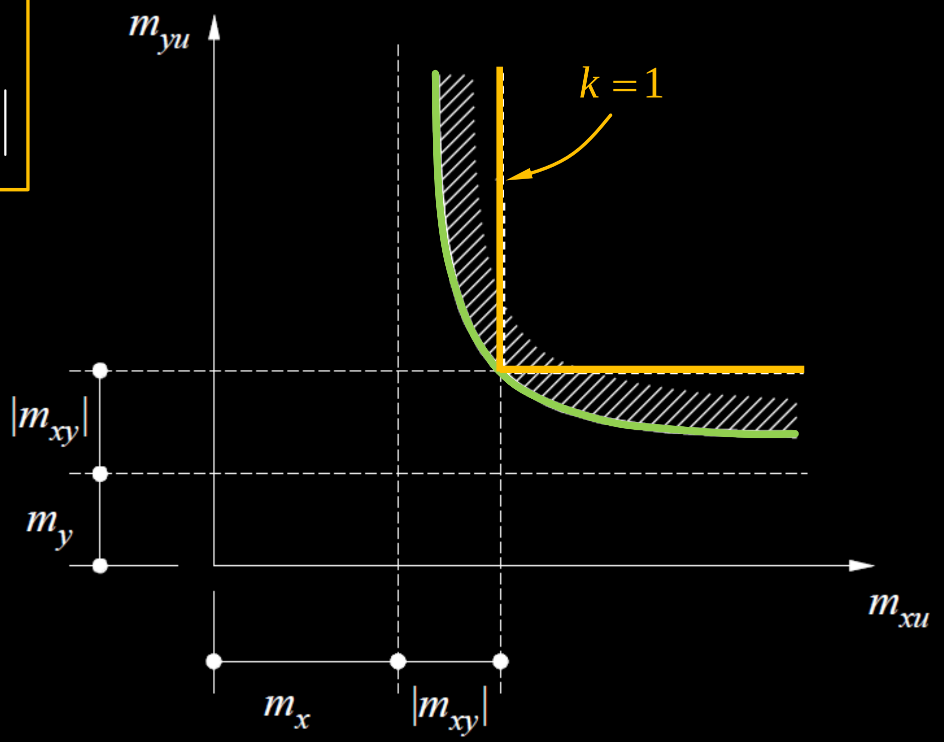
for positive bending moments:

$$\boxed{\begin{aligned} m_{x,u} &\geq m_x + k \cdot \left|m_{xy}\right| \\ m_{y,u} &\geq m_y + \frac{1}{k} \cdot \left|m_{xy}\right| \end{aligned}}$$

for negative bending moments:

$$\boxed{\begin{aligned} m'_{x,u} &\geq -m_x + k' \cdot \left|m_{xy}\right| \\ m'_{y,u} &\geq -m_y + \frac{1}{k'} \cdot \left|m_{xy}\right| \end{aligned}}$$

The parameter *k* can be freely selected and the reinforcement can be designed directly. If *k* = 1, the linearised yield condition follows, which is also used by many FE programs.



Slabs - Yield conditions

Design moments

The normal moment yield condition in parametric form: with $k = \left|\tan\varphi_u\right|$ and with $k' = \left|\tan\varphi'_u\right|$

The resulting design moments:

for positive bending moments:

$$\boxed{\begin{aligned} m_{x,u} &\geq m_x + k \cdot \left|m_{xy}\right| \\ m_{y,u} &\geq m_y + \frac{1}{k} \cdot \left|m_{xy}\right| \end{aligned}}$$

for negative bending moments:

$$\boxed{\begin{aligned} m'_{x,u} &\geq -m_x + k' \cdot \left|m_{xy}\right| \\ m'_{y,u} &\geq -m_y + \frac{1}{k'} \cdot \left|m_{xy}\right| \end{aligned}}$$

Dito, with notations according to SIA 262:

$$\boxed{\begin{aligned} m_{x,Rd} &\geq m_{x,d} + k \cdot \left|m_{xy,d}\right| \\ m_{y,Rd} &\geq m_{y,d} + \frac{1}{k} \cdot \left|m_{xy,d}\right| \end{aligned}}$$

$$\boxed{\begin{aligned} m'_{x,Rd} &\geq -m_{x,d} + k' \cdot \left|m_{xy,d}\right| \\ m'_{y,Rd} &\geq -m_{y,d} + \frac{1}{k'} \cdot \left|m_{xy,d}\right| \end{aligned}}$$

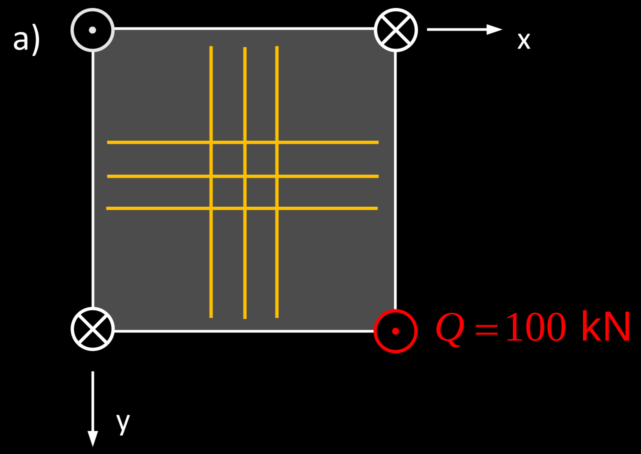
NB: For several loads or load combinations the required bending resistance $(m_x, m_y)_{Rd}$ should be determined for concomitant internal forces $(m_x, m_y, m_{xy})_d$, i.e., stress resultants obtained for the same load combination. The determination of the required bending resistances $(m_x, m_y)_{Rd}$ implemented in many FE programs from separately determined "limit values" for non-associated $m_{x,d}$, $m_{y,d}$ and $m_{xy,d}$ is often strongly on the safe side.

Slabs - Yield conditions

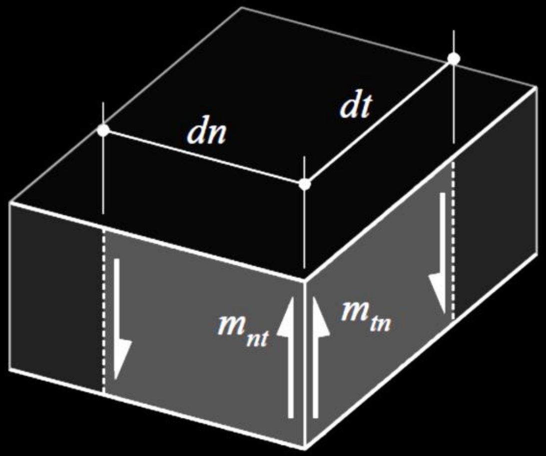
In-class exercise

Given: Square slab supported at 3 corners with side length l, acting corner force Q = 100 kN

Desired: Design moments for reinforcement in coordinate direction.



Hint: from the boundary shear force $V_t = -m_{tn}$, one gets the corner forces 2 m_{tn}



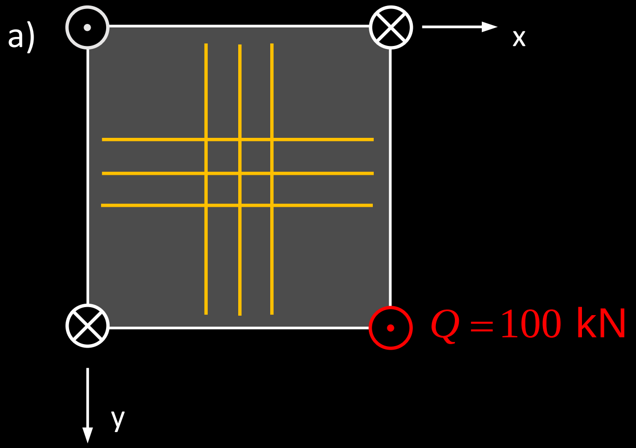
corner force

Slabs - Yield conditions

Design moments - Example

Given: Square slab supported at 3 corners with side length l, acting corner force Q = 100 kN

Desired: Design moments for reinforcement in coordinate direction and at 45° to it



Action: Corner force $2m_{xy} = Q = 100$ kN
(→ pure twisting with respect to the reinforcement directions (*x,y*))

$$m_x = m_y = 0$$

$$m_{xy} = 50 \text{ kN}$$

Linearised yield conditions (k = 1):

$$m_{x,u} \geq m_x + k \cdot |m_{xy}| = 0 + 50 = 50 \text{ kN} \qquad m_{y,u} \geq m_y + \frac{1}{k} \cdot |m_{xy}| = 0 + 50 = 50 \text{ kN}$$

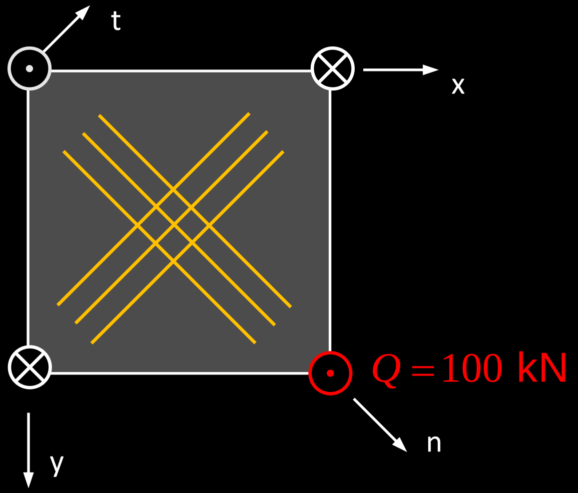
$$m'_{x,u} \geq -m_x + k' \cdot |m_{xy}| = 0 + 50 = 50 \text{ kN} \qquad m'_{y,u} \geq -m_y + \frac{1}{k'} \cdot |m_{xy}| = 0 + 50 = 50 \text{ kN}$$

→ All four reinforcement layers (top and bottom in *x*- and *y*-direction) have to be dimensioned for $m_u \geq 50$ kN

Slabs - Yield conditions

Design moments - Example

b) Rotation of the reinforcement by 45° to the *n*-*t*-direction



Actions: $\varphi = 45°$

(Reinforcement arranged in principal moment directions!)

$$m_n = m_x \cos^2 \varphi + m_y \sin^2 \varphi + m_{xy} \sin 2\varphi = m_{xy} = 50 \text{ kN}$$

$$m_t = m_x \sin^2 \varphi + m_y \cos^2 \varphi - m_{xy} \sin 2\varphi = -m_{xy} = -50 \text{ kN}$$

$$m_{nt} = \left(m_y - m_x\right) \sin \varphi \cos \varphi + m_{xy} \cos 2\varphi = 0$$

Linearised yield conditions:

$$m_{n,u} \geq m_n + k \cdot \left|m_{nt}\right| = 50 + 0 = 50 \text{ kN}$$

$$m_{t,u} \geq m_t + \frac{1}{k} \cdot \left|m_{nt}\right| = -50 + 0 = -50 \text{ kN} \rightarrow 0$$

$$m'_{n,u} \geq -m_n + k' \cdot \left|m_{nt}\right| = -50 + 0 = -50 \text{ kN} \rightarrow 0$$

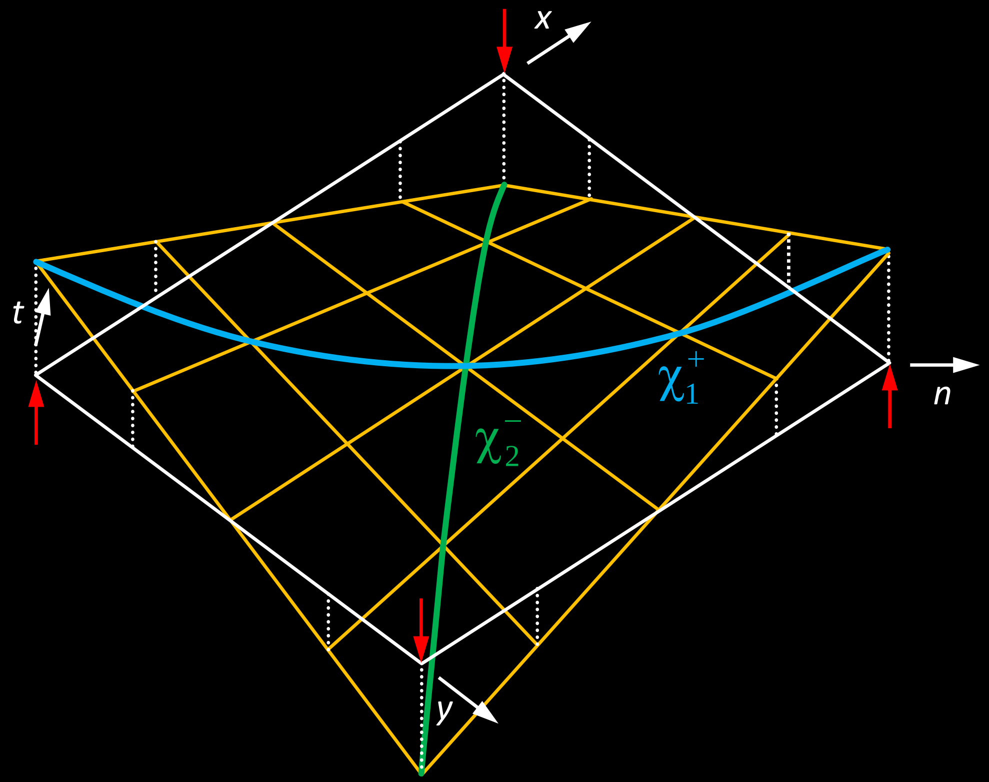
$$m'_{t,u} \geq -m_t + \frac{1}{k'} \cdot \left|m_{nt}\right| = 50 + 0 = 50 \text{ kN}$$

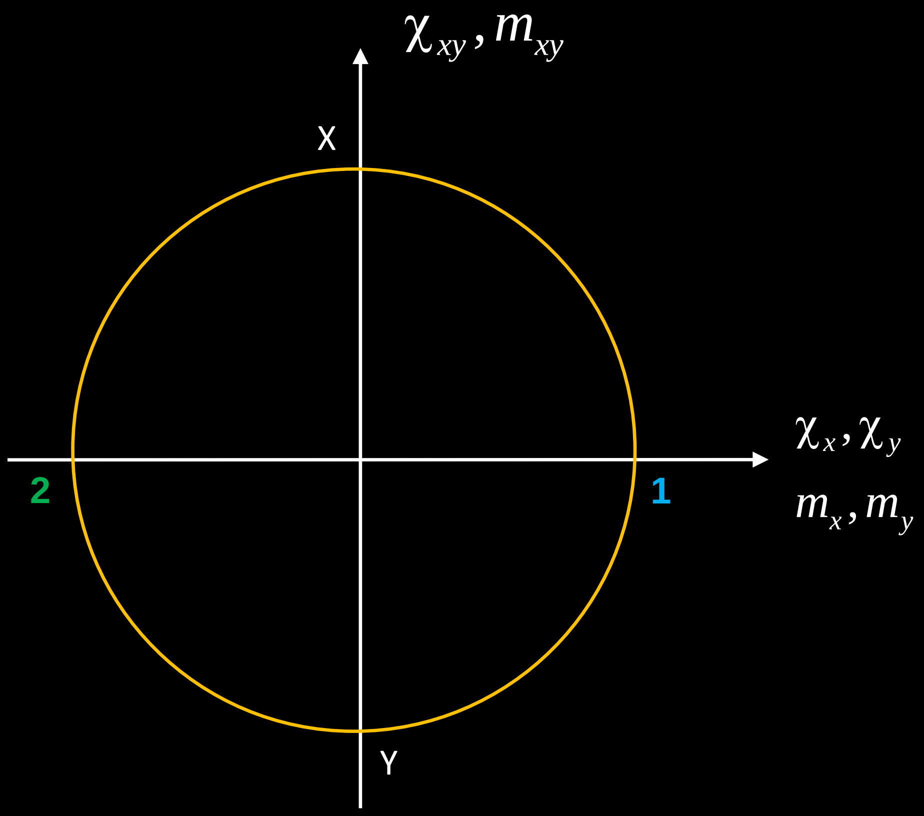
→ Half the amount of reinforcement is sufficient for the reinforcement in the principal moment direction: lower reinforcement in the *n*-direction and upper reinforcement in the *t*-direction require each: $m_u \geq 50$ kN (negative design moments: no reinforcement required).

→ "Trajectory reinforcement" optimal, but rarely practicable (complicated reinforcement layout, principal directions change due to changing actions)

Slabs - Yield conditions

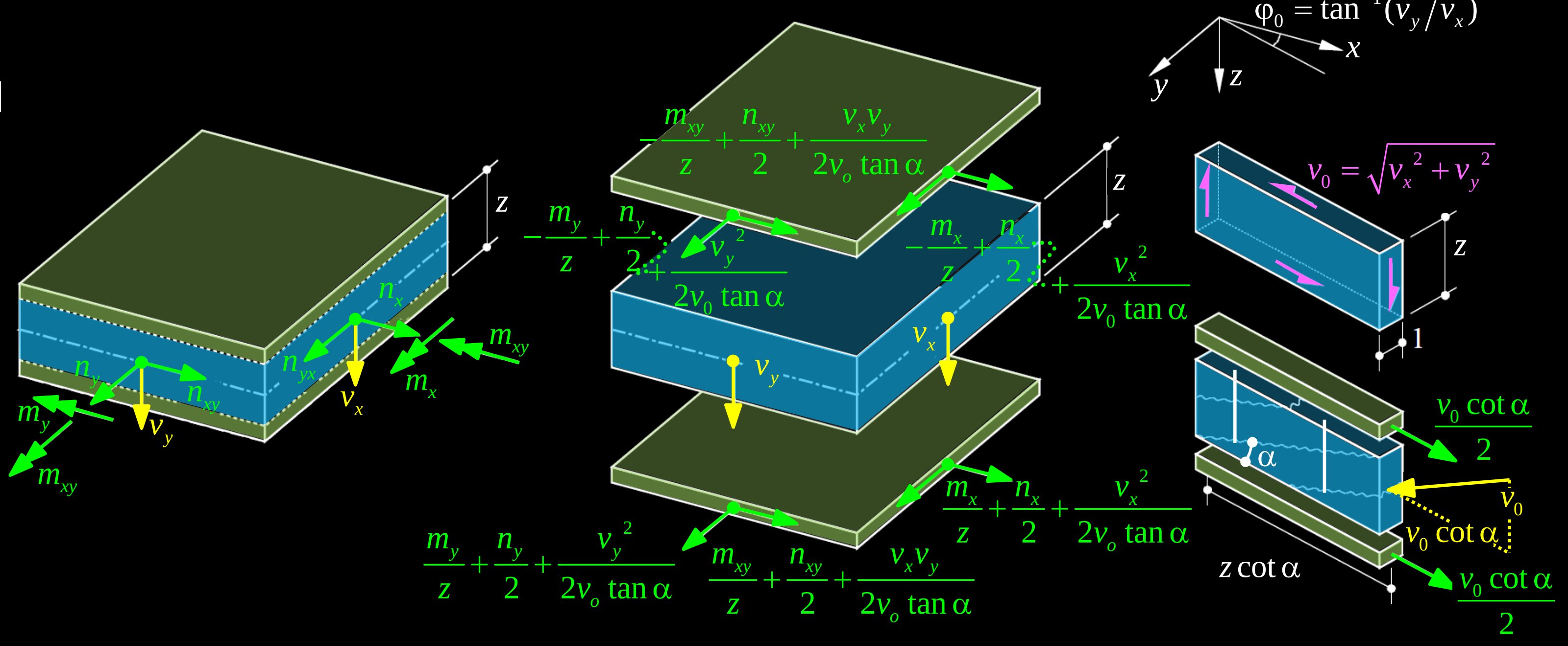
Pure twisting $\chi_{xy}\left(\chi_x = \chi_y = 0\right)$





Slabs - Yield conditions

Sandwich model



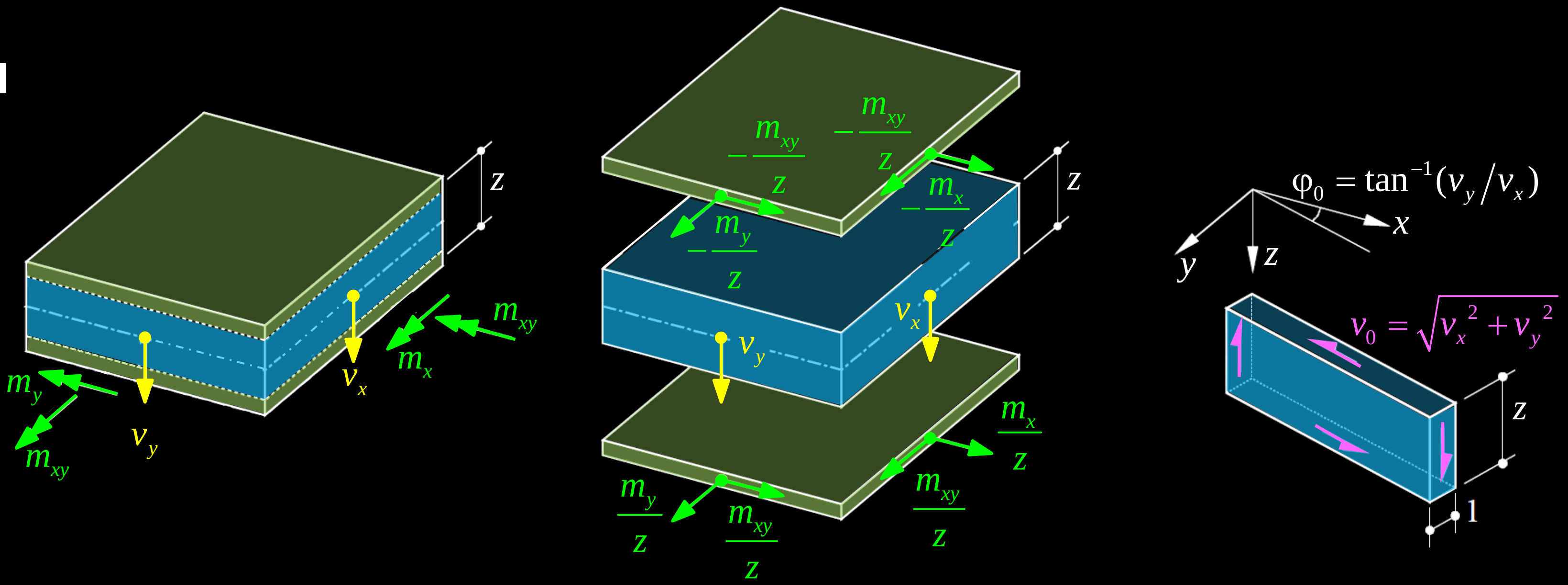
Equilibrium solution for general shell loading (statically admissible stress state):

- Sandwich covers carry bending and twisting moments (substituted by statically equivalent force couples ±*m*/*z* in bottom and top cover) as well as possible membrane forces (substituted by statically equivalent forces *n*/2 in each cover)
  - → In-plane loading of each cover, treatment as membrane elements with corresponding reinforcement, dimensioning with yield conditions for membrane elements
  - → Suitable for the design of generally loaded shell elements (8 stress resultants)
- Sandwich core absorbs shear forces
  - → Sandwich core absorbs principal transverse force v_0 in the direction φ_0 (see transverse shear in slabs)

NB: High membrane (compressive) forces: core can also be used for this (note interaction with *v*)

Slabs - Yield conditions

Sandwich model



→ Slabs under pure bending without shear reinforcement:
$n_x = n_y = n_{xy} = 0,\ v_{0d} \leq v_{Rd} = k_d \tau_{cd} d_v$

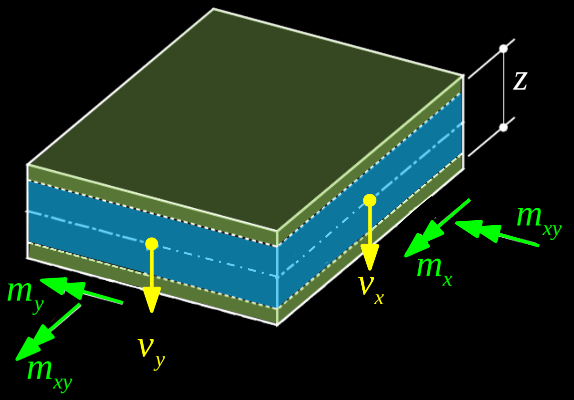
→ Terms with n_x, n_y, n_{xy} are zero

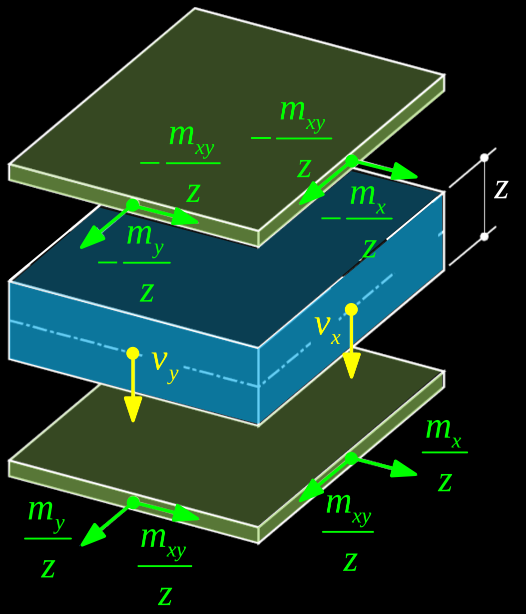
→ Terms with v_x, v_y are omitted if an uncracked core is assumed.

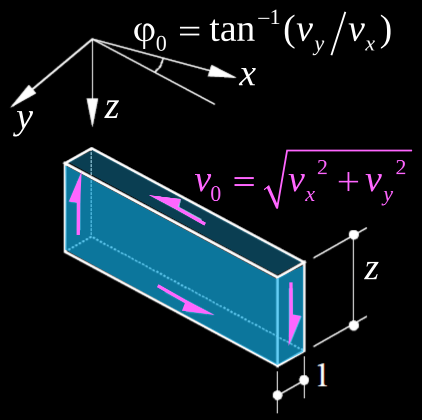
→ Yield conditions for slabs based on the sandwich model = simplification of the general case of a shell element with eight stress resultants (slab: only bending and twisting moments considered, consideration of transverse (slab) shear forces → see shear force in slabs)

Slabs - Yield conditions

Sandwich model







→ Reinforcement of the sandwich covers = yield conditions for slabs according to static method:

$$a_{sx} f_{sd} \geq \frac{m_x}{z} + k \left| \frac{m_{xy}}{z} \right|$$

$$a_{sy} f_{sd} \geq \frac{m_y}{z} + \frac{1}{k} \left| \frac{m_{xy}}{z} \right|$$

$$a_{sx}^{'} f_{sd} \geq -\frac{m_x}{z} + k^{'} \left| -\frac{m_{xy}}{z} \right|$$

$$a_{sy}^{'} f_{sd} \geq -\frac{m_y}{z} + \frac{1}{k^{'}} \left| -\frac{m_{xy}}{z} \right|$$

i.e. $m_{xu} \geq m_x + k|m_{xy}| \qquad m_{yu} \geq m_y + k^{-1}|m_{xy}|$

$m'_{xu} \geq -m_x + k'|m_{xy}| \qquad m'_{yu} \geq -m_y + k'^{-1}|m_{xy}|$

and by multiplication follows:

$$\left(\frac{m_{xy}}{z}\right)^2 - \left(\frac{m_{xu}}{z} - \frac{m_x}{z}\right)\left(\frac{m_{yu}}{z} - \frac{m_y}{z}\right) = 0$$

$$\left(\frac{m_{xy}}{z}\right)^2 - \left(\frac{m_{xu}^{'}}{z} + \frac{m_x}{z}\right)\left(\frac{m_{yu}^{'}}{z} + \frac{m_y}{z}\right) = 0$$

$m_{xu} = z a_{sx} f_{sd} \qquad m_{yu} = z a_{sy} f_{sd}$

$m'_{xu} = z a'_{sx} f_{sd} \qquad m'_{yu} = z a'_{sy} f_{sd}$

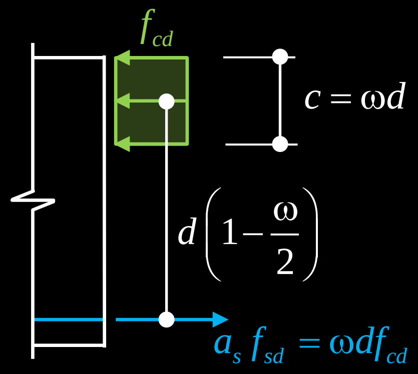
Condition for «Regime 1»
(not from normal moment yield criterion):

$$f_{cd} z t_{inf} \geq m_{xu} - m_x + m_{yu} - m_y$$

$$f_{cd} z t_{sup} \geq m_{xu}^{'} + m_x + m_{yu}^{'} + m_y$$

Slabs - Yield conditions

Pure twisting - Investigation with sandwich model (lower limit value)



Isotropic reinforcement: $m_{x,u}=m_{y,u}=m_u=m'_{x,u}=m'_{y,u}$

$$m_u=a_s f_{sd}\left(d-\frac{a_s f_{sd}}{2 f_{cd}}\right)=d^2 f_{cd}\omega\left(1-\frac{\omega}{2}\right)$$

- Normal moment yield condition: $m_{xy}=m_u$

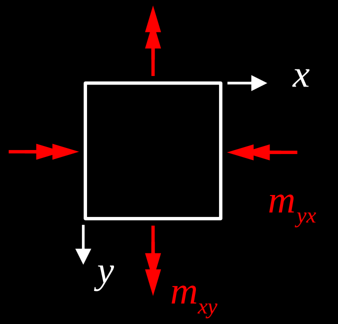
$$m_{xy}^2-(m_{x,u}-m_x)(m_{y,u}-m_y)=0 \quad \text{with } m_x, m_y \;=\; 0$$

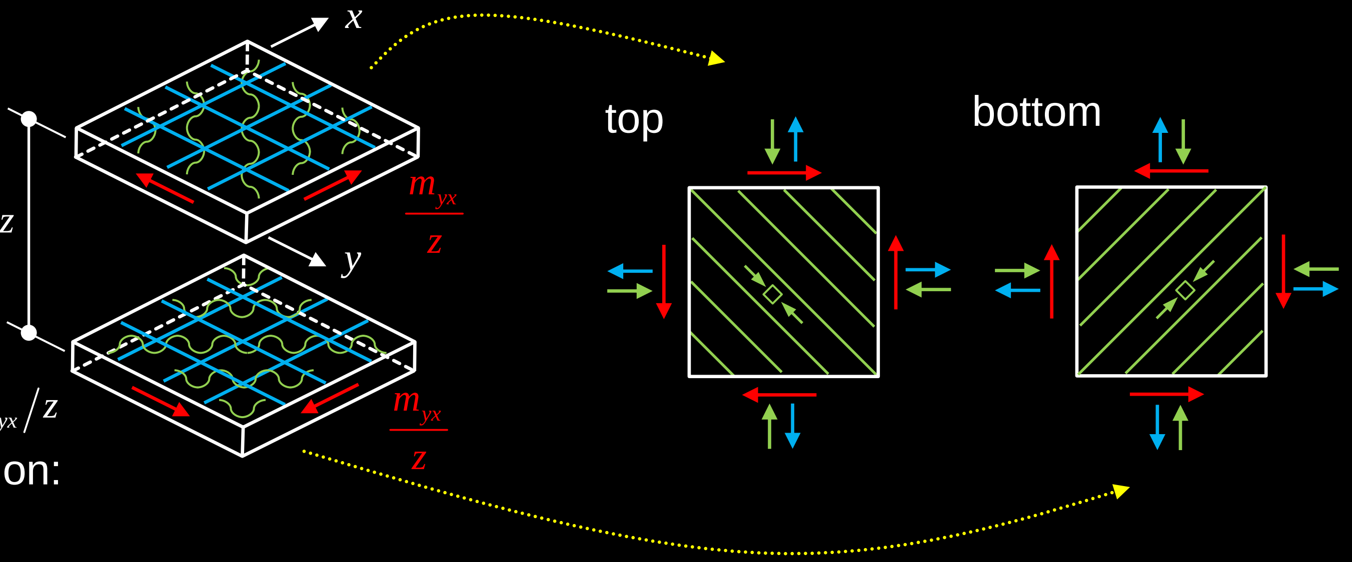
$$\rightarrow m_{xy}=\sqrt{m_{x,u}m_{y,u}}=m_u \qquad\qquad \text{analogous for } m'$$

Slabs - Yield conditions

Pure twisting - Investigation with sandwich model (lower limit value)

- Lower limit value

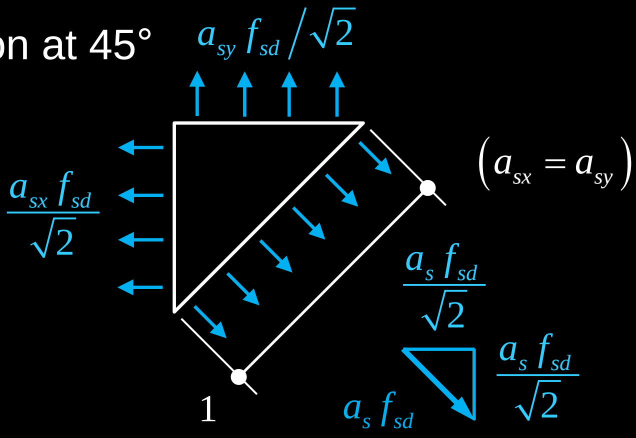




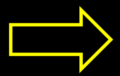
$$\rightarrow a_{sx} f_{sd} = a_{sy} f_{sd} = a'_{sx} f_{sd} = a'_{sy} f_{sd} = m_{yx}/z$$

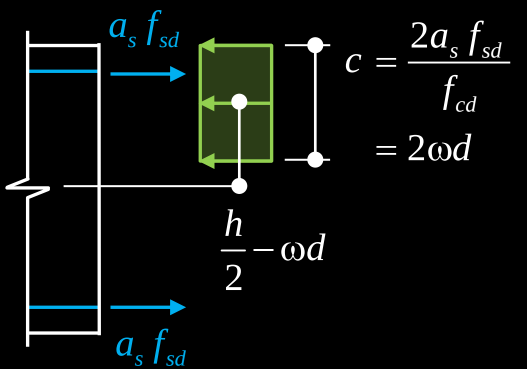
All reinforcements yield in tension:

Section at 45°



$$\left(a_{sx} = a_{sy}\right)$$



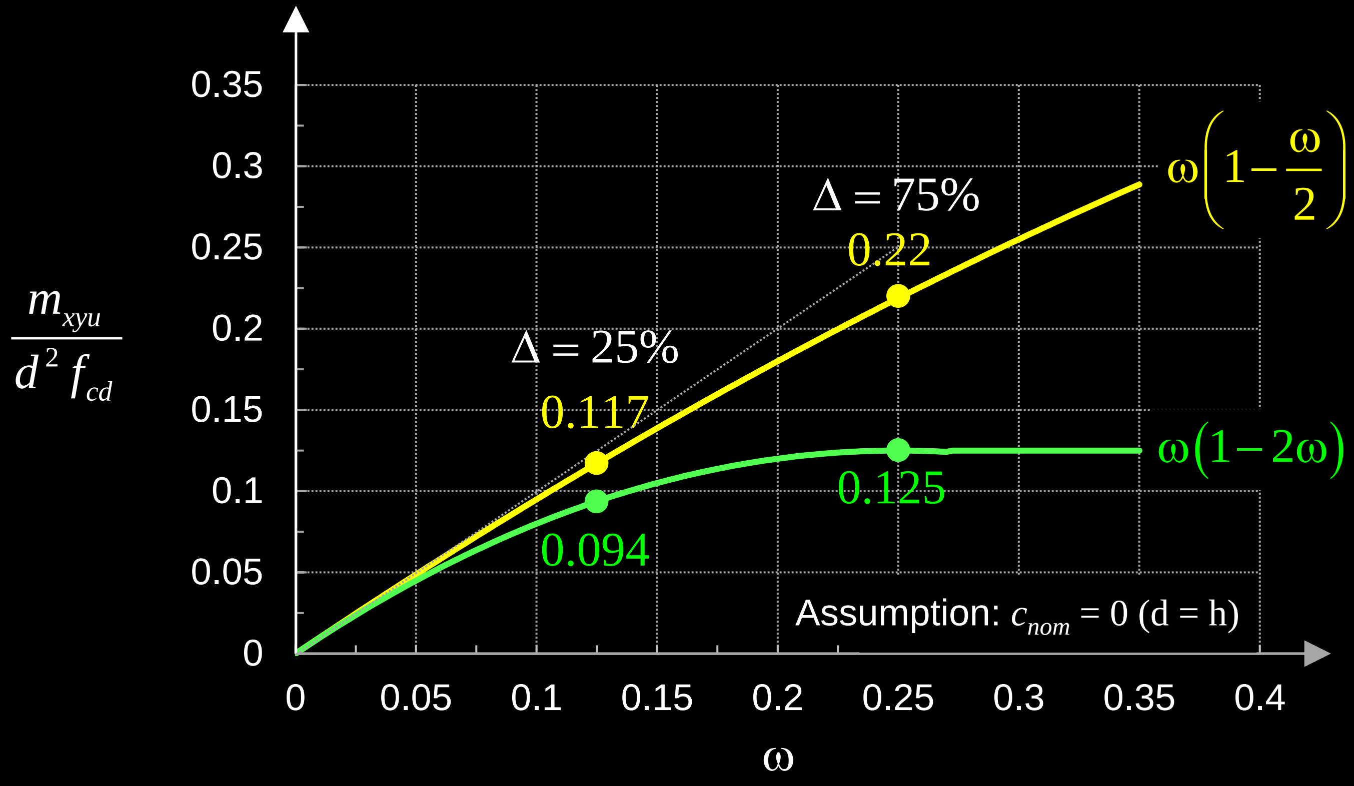


$$c = \frac{2a_s f_{sd}}{f_{cd}} = 2\omega d$$

$$m_u = 2\omega d f_{cd}\left(\frac{h}{2} - \omega d\right) = d^2 f_{cd}\omega\left(\frac{h}{d} - 2\omega\right) < d^2 f_{cd}\omega\left(1 - \frac{\omega}{2}\right)$$

Slabs - Yield conditions

Pure twisting - Investigation with sandwich model (lower limit value)



Normal moment yield condition

Twisting moment resistance

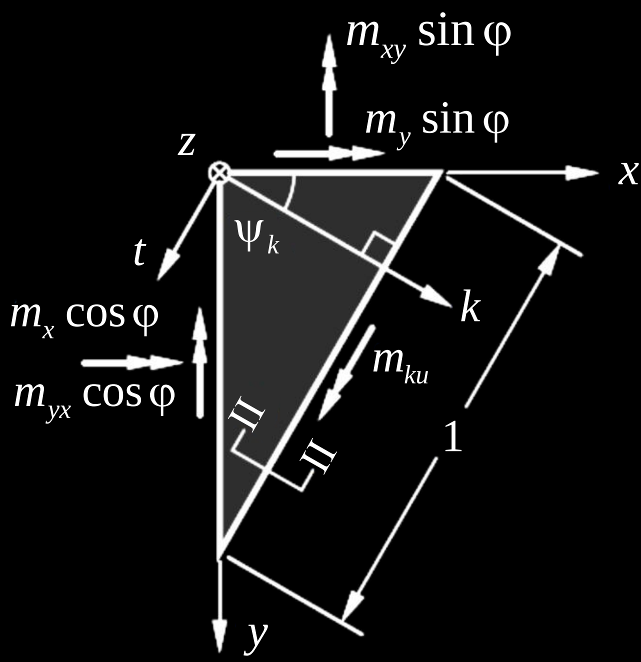
Corner supports with large twisting moments → Caution!

Slabs - Yield conditions

Yield conditions for skew reinforcement

Superposition of the bending resistances of *k* reinforcement layers in the reinforcement directions ψ_k

(Transformation of all $\{m_k = m_{ku},\ m_t = 0\}$ in the directions x,y):



$$\mu_x = \sum_{k-1}^{r} m_{ku}\cos^2\psi_k \qquad \mu'_x = \sum_{k-1}^{r} m'_{ku}\cos^2\psi_k$$

$$\mu_y = \sum_{k-1}^{r} m_{ku}\sin^2\psi_k \qquad \mu'_y = \sum_{k-1}^{r} m'_{ku}\sin^2\psi_k$$

$$\mu_{xy} = \sum_{k-1}^{r} m_{ku}\sin\psi_k\cos\psi_k \qquad \mu'_{xy} = \sum_{k-1}^{r} m'_{ku}\sin\psi_k\cos\psi_k$$

Normal moment yield condition for skew reinforcement:

$$m_{au}(\varphi) \approx \sum_{k=1}^{r} m_{ku}\cos^2(\varphi-\psi_k) = \mu_x\cos^2\varphi + \mu_y\sin^2\varphi + 2\mu_{xy}\sin\varphi\cos\varphi$$

$$m'_{au}(\varphi) \approx \sum_{k=1}^{r} m'_{ku}\cos^2(\varphi-\psi_k) = \mu'_x\cos^2\varphi + \mu'_y\sin^2\varphi + 2\mu'_{xy}\sin\varphi\cos\varphi$$

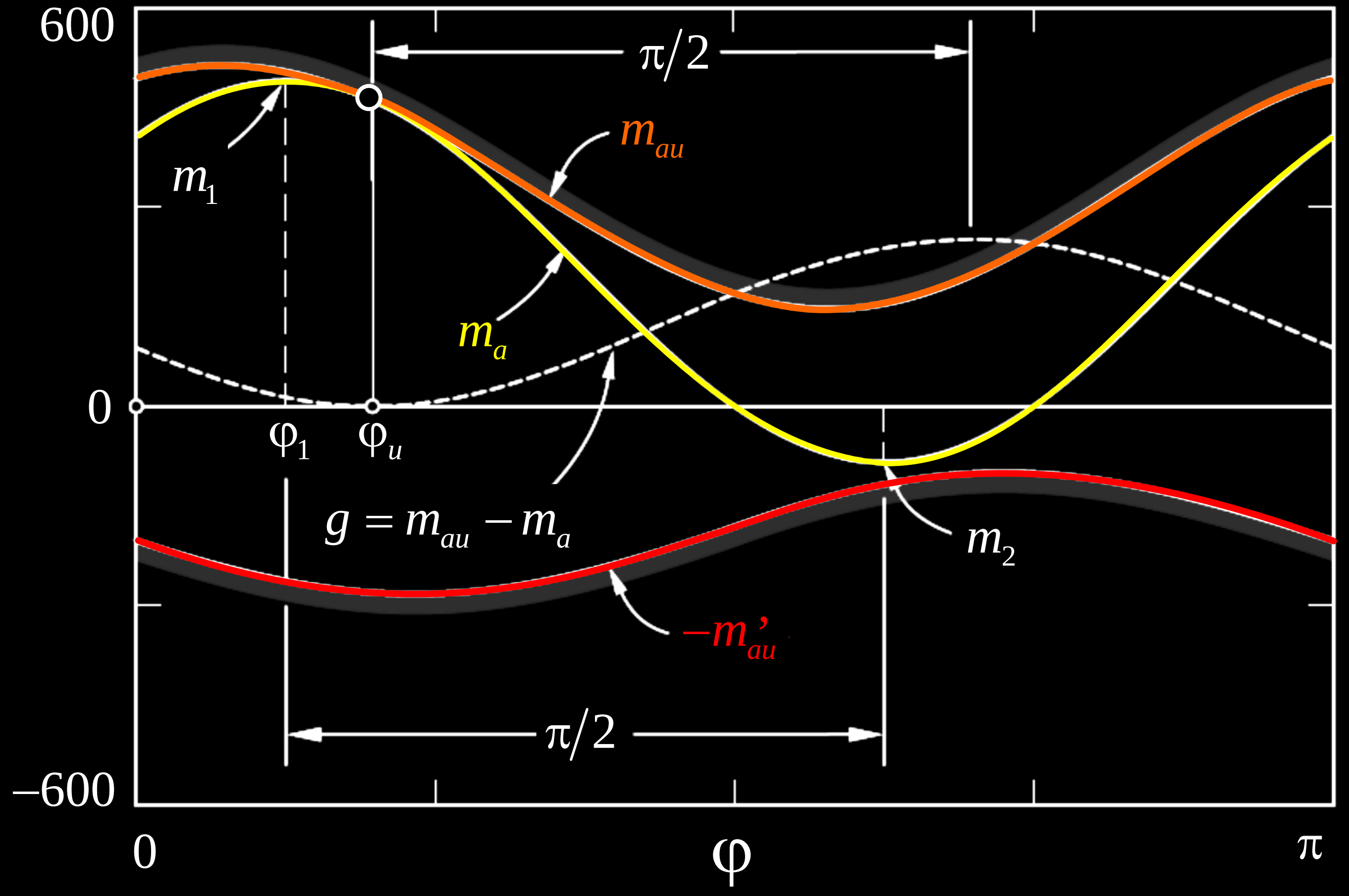
('≈' since different compression zone heights → no compatible mechanism. But compression fields in the concrete are not orthogonal → f_{cd} exceeded, thus no clear lower/upper limit value. For not too high reinforcement ratios however very good approximation)

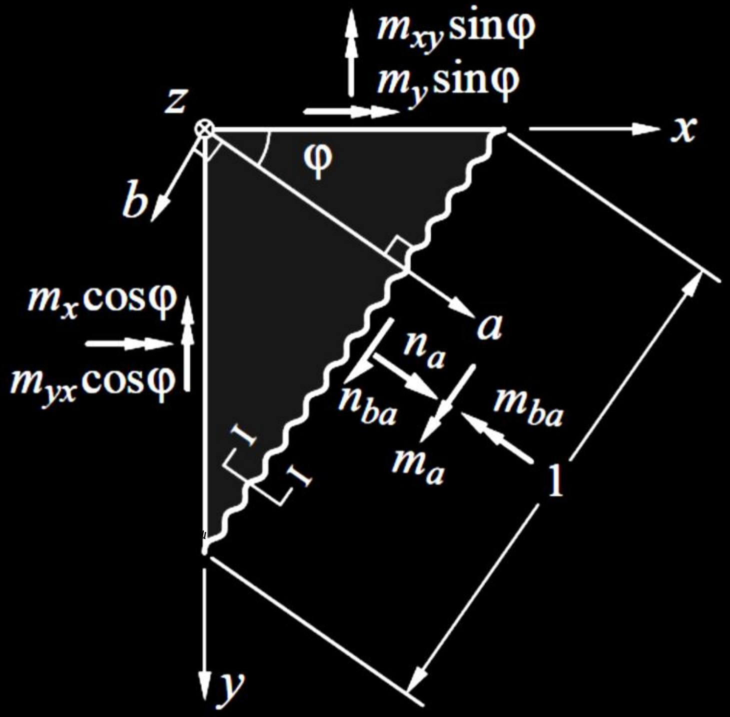
Check condition $m'_{au}(\varphi) \le m_a(\varphi) \le m_{au}(\varphi)$ in all directions φ (see next slide)

Slabs - Yield conditions

Yield conditions for skew reinforcement

Check condition in all directions φ: $m'_{au}(\varphi) \le m_a(\varphi) \le m_{au}(\varphi)$





[Seelhofer (2009)]

Slabs - Yield conditions

Yield conditions for skew reinforcement

Superposition of the bending resistances of *k* reinforcement layers in the reinforcement directions ψ_k

(Transformation of all $\{m_k = m_{ku},\ m_t = 0\}$ in the directions x,y):

$$\mu_x = \sum_{k-1}^{r} m_{ku} \cos^2 \psi_k \qquad \mu_x^{'} = \sum_{k-1}^{r} m_{ku}^{'} \cos^2 \psi_k$$

$$\mu_y = \sum_{k-1}^{r} m_{ku} \sin^2 \psi_k \qquad \mu_y^{'} = \sum_{k-1}^{r} m_{ku}^{'} \sin^2 \psi_k$$

$$\mu_{xy} = \sum_{k-1}^{r} m_{ku} \sin \psi_k \cos \psi_k \qquad \mu_{xy}^{'} = \sum_{k-1}^{r} m_{ku}^{'} \sin \psi_k \cos \psi_k$$

Bending resistance in the direction φ:

$$m_{au}(\varphi) \approx \sum_{k=1}^{r} m_{ku} \cos^2(\varphi - \psi_k) = \mu_x \cos^2\varphi + \mu_y \sin^2\varphi + 2\mu_{xy} \sin\varphi \cos\varphi$$

$$m_{au}^{'}(\varphi) \approx \sum_{k=1}^{r} m_{ku}^{'} \cos^2(\varphi - \psi_k) = \mu_x^{'} \cos^2\varphi + \mu_y^{'} \sin^2\varphi + 2\mu_{xy}^{'} \sin\varphi \cos\varphi$$

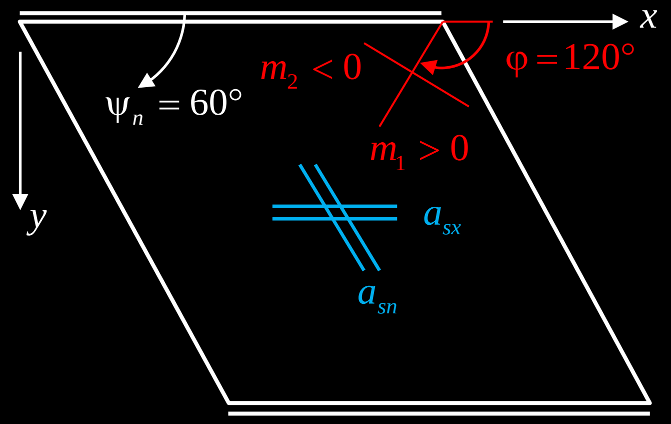
Check condition $m_{au}^{'}(\varphi) \leq m_a(\varphi) \leq m_{au}(\varphi)$ in all directions φ

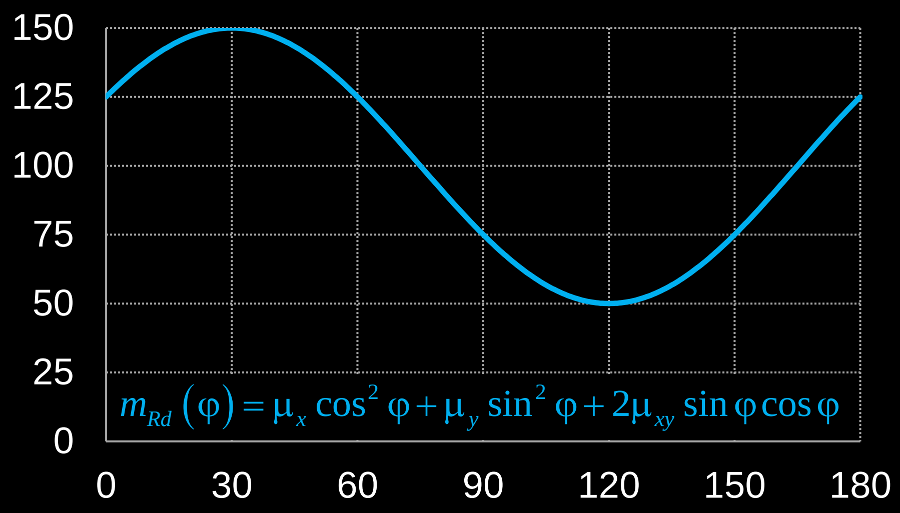
→ Normal moment yield criterion for skew reinforcement:

$$\boxed{\begin{aligned} Y &= \left(\mu_{xy} - m_{xy}\right)^2 - \overbrace{\left(\mu_x - m_x\right)}^{\geq 0}\overbrace{\left(\mu_y - m_y\right)}^{\geq 0} = 0 \\ Y' &= \left(\mu'_{xy} + m_{xy}\right)^2 - \underbrace{\left(\mu'_x + m_x\right)}_{\geq 0}\underbrace{\left(\mu'_y + m_y\right)}_{\geq 0} = 0 \end{aligned}}$$

Slabs - Yield conditions

Example of skew reinforcement





$$m_{Rdx} = 100 \text{ kNm/m}$$
$$m_{Rdn} = 100 \text{ kNm/m}$$
$$\psi_n = 60°$$

$$\mu_x = m_{Rdx} \cdot \cos^2 0° + m_{Rdn} \cdot \cos^2 60° = 125 \text{ kNm/m}$$
$$\mu_y = m_{Rdx} \cdot \sin^2 0° + m_{Rdn} \cdot \sin^2 60° = 75 \text{ kNm/m}$$
$$\mu_{xy} = m_{Rdx} \cdot \sin 0° \cos 0° + m_{Rdn} \cdot \sin 60° \cos 60° = \sqrt{3} \cdot 25 = 43.3 \text{ kNm/m}$$

$$\varphi = 120°: \; m_{Rdmin} = 50 \text{ kNm/m}$$
$$\varphi = 30°: \; m_{Rdmax} = 150 \text{ kNm/m}$$

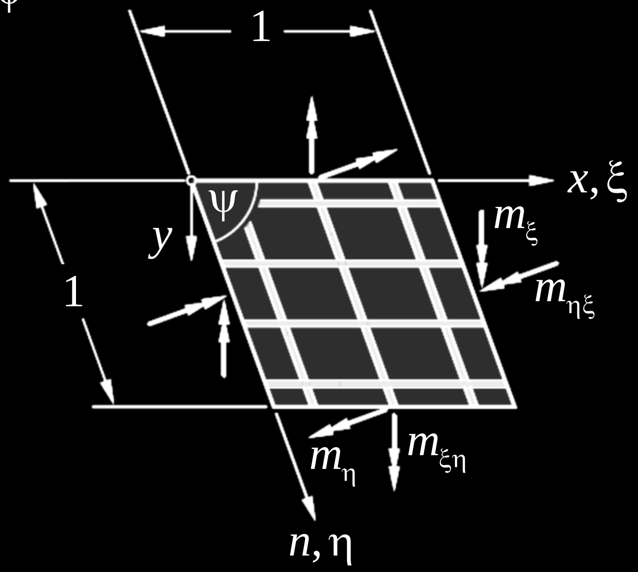
Maxima and minima of the bending resistances do not occur in the reinforcement directions.
Rather, a minimum occurs in the direction of the bisector of the obtuse angle. The resistance is significantly reduced even with slight skewness.

Slabs - Yield conditions

Yield conditions for skew reinforcement

Using oblique coordinates, design equations can be formulated (as with membrane elements):

$$m_\xi = m_x \sin\psi + m_y \cos\psi \cot\psi - 2m_{xy} \cos\psi \qquad m_\eta = m_y / \sin\psi \qquad m_{\xi\eta} = m_{\eta\xi} = m_{xy} - m_y \cot\psi$$



The normal moment yield criterion in oblique coordinates is:

(with conditions)

$$Y = m_{\xi\eta}^2 - \left(m_{xu} \sin\psi - m_\xi\right)\left(m_{nu} \sin\psi - m_\eta\right) = 0 \qquad Y' = m_{\xi\eta}^2 - \left(m_{xu}' \sin\psi + m_\xi\right)\left(m_{nu}' \sin\psi + m_\eta\right) = 0$$

$$-m_{xu}' \sin\psi \le m_\xi \le m_{xu} \sin\psi \qquad -m_{nu}' \sin\psi \le m_\eta \le m_{nu} \sin\psi$$

Notation in parametric form
→ direct dimensioning possible:

(Parameters *k* and *k*' freely selectable, minimum reinforcement results for *k* = *k*' = 1)

$$m_{xu} \ge \frac{1}{\sin\psi}\left(m_\xi + k\left|m_{\xi\eta}\right|\right) \qquad m_{nu} \ge \frac{1}{\sin\psi}\left(m_\eta + k^{-1}\left|m_{\xi\eta}\right|\right) \qquad m_{xu}' \ge \frac{1}{\sin\psi}\left(-m_\xi + k'\left|m_{\xi\eta}\right|\right) \qquad m_{nu}' \ge \frac{1}{\sin\psi}\left(-m_\eta + \left(k'\right)^{-1}\left|m_{\xi\eta}\right|\right)$$

$$k = \left|\sin\psi \tan\varphi_u + \cos\psi\right| \qquad k' = \left|\sin\psi \tan\varphi_u' + \cos\psi\right|$$

[Seelhofer (2009)]

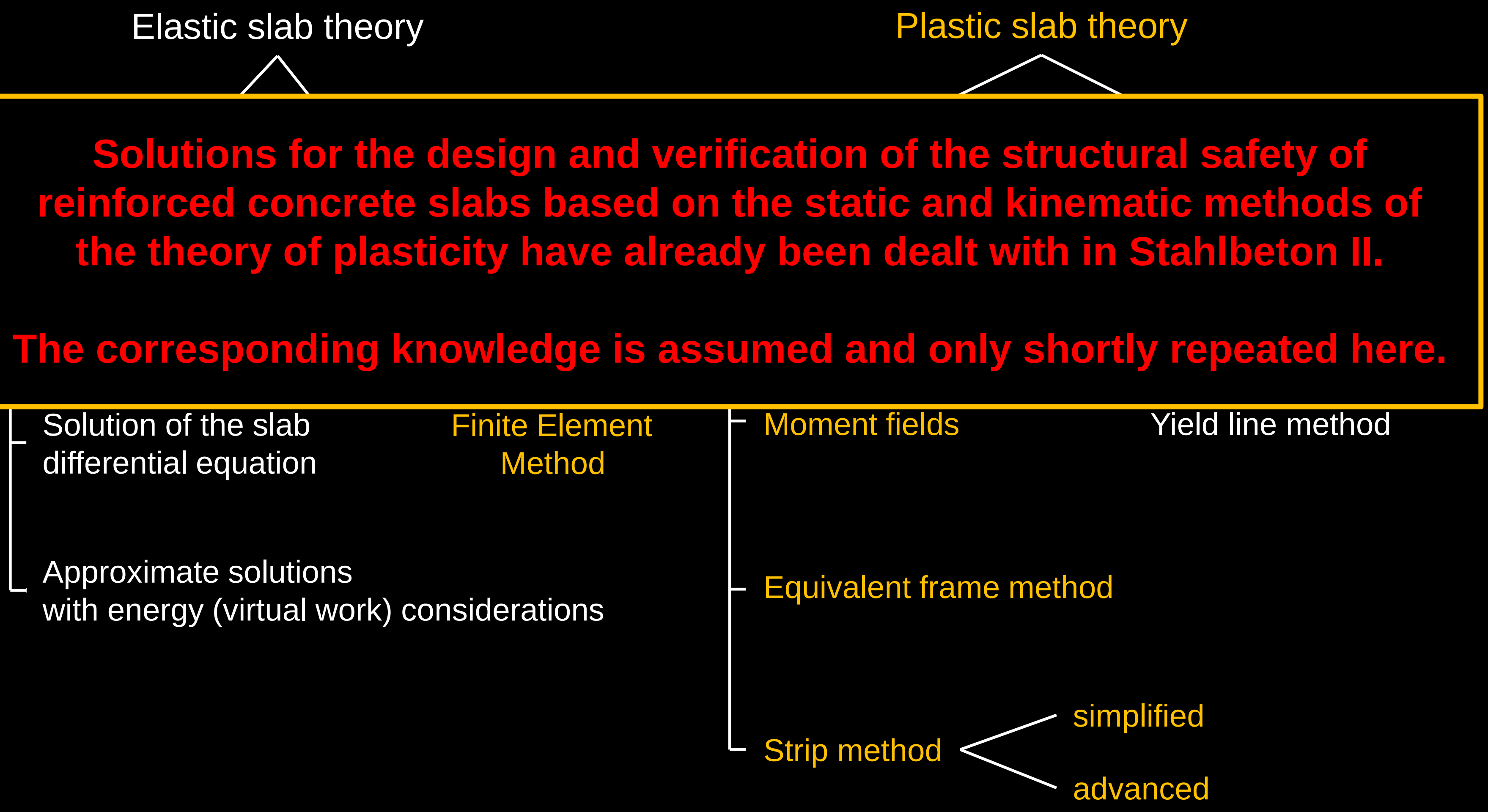
5 Slabs

In-depth study and additions to Stahlbeton II (Chapter 7.2)

5.3 Equilibrium solutions

Slabs - Equilibrium solutions

Structural analysis / Calculation methods - Overview



Elastic slab theory

Plastic slab theory

Solutions for the design and verification of the structural safety of reinforced concrete slabs based on the static and kinematic methods of the theory of plasticity have already been dealt with in Stahlbeton II.

The corresponding knowledge is assumed and only shortly repeated here.

- Solution of the slab differential equation
- Approximate solutions with energy (virtual work) considerations

Finite Element Method

- Moment fields
- Equivalent frame method
- Strip method
  - simplified
  - advanced

Yield line method

Slabs - Equilibrium solutions

Overview

Equilibrium solutions are based on the lower or static limit theorem of the theory of plasticity.

Requirements:	→ statically admissible stress state (equilibrium and static boundary conditions satisfied)
	→ yield conditions not violated anywhere

Determination of statically admissible stress states:

• Elastic slab theory:	In addition to equilibrium and static boundary conditions, the elastic compatibility conditions are also satisfied here. The finite element method can be used to treat cases with any geometry and load (the most common method today). In addition, there are various textbooks with corresponding tables.
• Moment fields:	Combination of different moment fields for selected geometries and loads
• Strip method:	This method, which goes back to Hillerborg, assumes strip-shaped bending elements in two usually orthogonal directions (simple strip method). With the advanced strip method, concentrated forces can be treated with the aid of corresponding moment fields or load distribution elements.
• Equivalent frame method:	Global equilibrium solutions for flat and mushroom slabs (distribution of moments in transverse direction based on elastic solutions).

Slabs - Equilibrium solutions

Overview

Equilibrium solutions are particularly suitable for the design of slabs. If a slab is dimensioned according to these methods and if its deformation capacity is sufficient, its load-carrying capacity will in no case be less than the corresponding load.

The static method of the theory of plasticity ensures sufficient bending resistance. However, the influence of shear forces is not taken into account and must be investigated separately.

If a compatible failure mechanism is found for an equilibrium solution (see chapter yield line method), it corresponds to a complete solution according to the theory of plasticity. This results in the (theoretically) exact ultimate load.

Slabs - Equilibrium solutions

Simple strip method: Basics

→ Neglect the twisting moments, satisfy equilibrium conditions only with m_x and m_y
→ Divide the load q into the parts q_x and q_y ($q_{xy} = 0$)

$$\boxed{\frac{\partial^2 m_x}{\partial x^2} + 2\cancel{\frac{\partial^2 m_{xy}}{\partial x \partial y}} + \frac{\partial^2 m_y}{\partial y^2} + q = 0 \qquad \rightarrow q = q_x + q_y\,, \qquad \underbrace{\frac{\partial^2 m_x}{\partial x^2} = -q_x\,,}_{\text{beam in x-direction}} \qquad \underbrace{\frac{\partial^2 m_y}{\partial y^2} = -q_y}_{\text{beam in y-direction}}}$$

→ Total load q is thus carried by the beam load-bearing effect in *x*- and *y*-direction
→ Distribution of the load can be freely chosen.
→ In order to ensure sufficient deformation capacity and satisfactory behaviour in serviceability limit state, q_x and q_y should be chosen cautiously.
→ This also applies to the calculation of the individual (often hyperstatic) strips according to beam theory.

The idea of considering a slab as a group of beams orthogonal to each other was developed very early on. Marcus (1931) suggested that the distribution of the load should be such that the elastic deflections of the fictitious beams in the middle of the slab coincide (→ hint for selection of distributed load: per direction ~ L^{-4}).

Hillerborg showed that the strip method is an application of the lower limit theorem of the theory of plasticity and generalised the method.

Slabs - Additions

Advanced strip method: Load distribution elements

Load distribution elements are used to treat supports and concentrated loads with the strip method. These convert a point load into a uniformly distributed load or vice versa. They thus correspond to the solutions for point-supported slabs (in the middle) under uniform loads.

Supports: The load distribution elements are regarded as area bearings with uniform compression, which are loaded by indirectly supported strips or (usually) hidden beams. The bending resistances resulting from the beams are increased in order to account for the bending resistances required for load transfer in the column area (= load distribution element).

Individual loads: The individual loads are applied to the slab as uniformly distributed surface loads, which are transferred to the supports by strips or (usually) hidden beams. The resulting bending resistances of the strips are superimposed with the bending resistances required to convert the point load into an evenly distributed area load (= load distribution element).

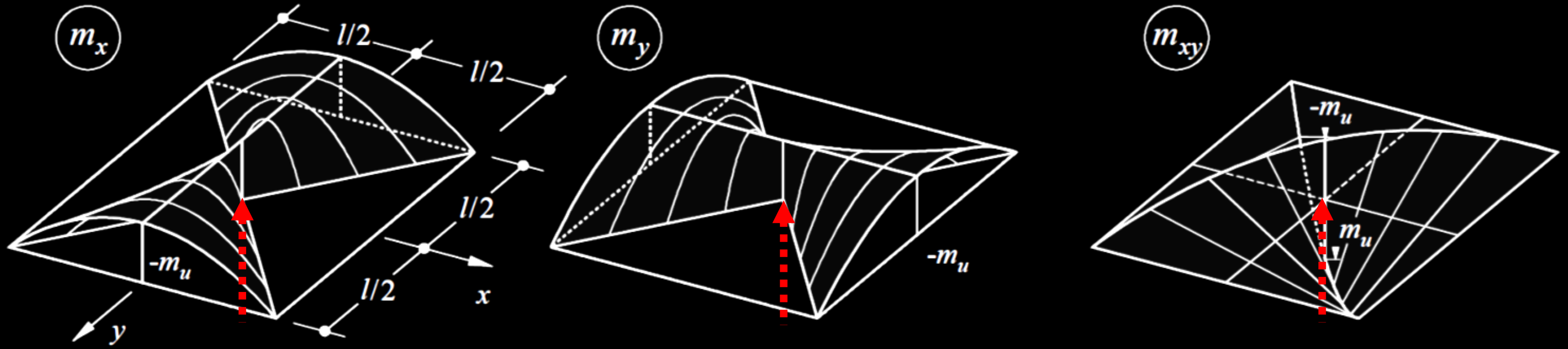
Slabs - Additions

Advanced strip method: Load distribution elements - Repetition moment fields

The moment fields below are suitable as "load distribution elements" for converting point loads into area loads.

If constant positive moments m_x und m_y are superimposed on them, the lower bound value for the load-carrying capacity of an infinitely extended flat slab under uniformly distributed load is determined with $m_{xu} = m_{yu} = m_u$ and $m'_{xu} = m'_{yu} = \lambda m_u$ (Marti 1981):

$$\boxed{q \geq 4(1+\lambda)\frac{m_u}{l^2}, \ \lambda = \frac{m'_{xu}}{m_u} = \frac{m'_{yu}}{m_u}}$$



$$m_x = 0$$

$$m_x = m_u\left(\frac{x^2}{y^2} - 1\right)$$

$$m_y = m_u\left(\frac{y^2}{x^2} - 1\right)$$

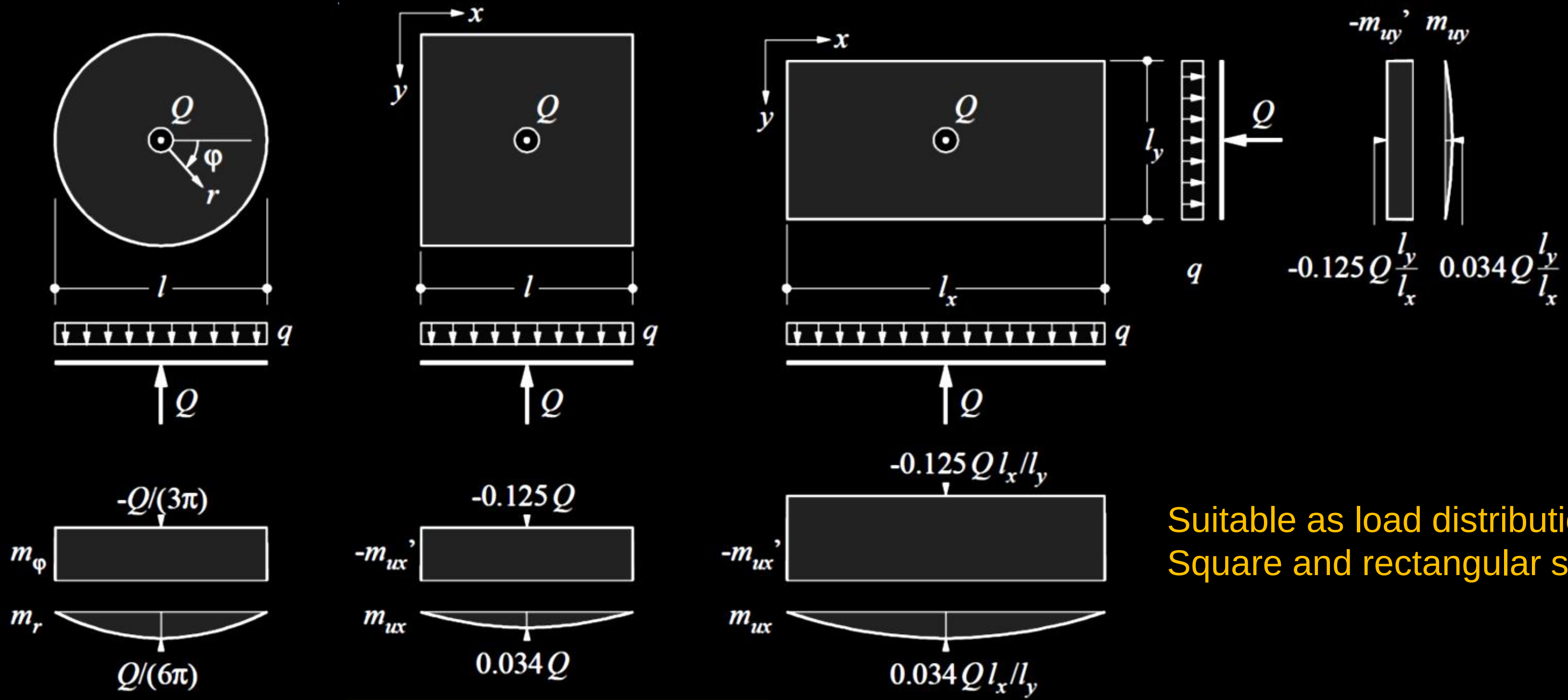
$$m_y = 0$$

$$m_{xy} = m_u\left(\frac{y}{x} - \frac{4xy}{l^2}\right) \quad \text{for } \left(x^2 > y^2\right)$$

$$m_{xy} = m_u\left(\frac{x}{y} - \frac{4xy}{l^2}\right) \quad \text{for } \left(x^2 < y^2\right)$$

Slabs - Additions

Advanced strip method: Load distribution elements



Suitable as load distribution elements:
Square and rectangular slabs

$$\left(m_u + m'_u\right) \geq \frac{Q}{2\pi}$$

$$m'_u \geq \frac{Q}{8}$$

$$m_u \geq \left(\frac{1}{2\pi} - \frac{1}{8}\right) Q = 0.034 \cdot Q$$

$$m'_{x,u} \geq \frac{Q}{8}\frac{l_x}{l_y} \qquad m_{x,u} \geq 0.034 \cdot q l_x^{\,2} = 0.034 \cdot Q \frac{l_x}{l_y}$$

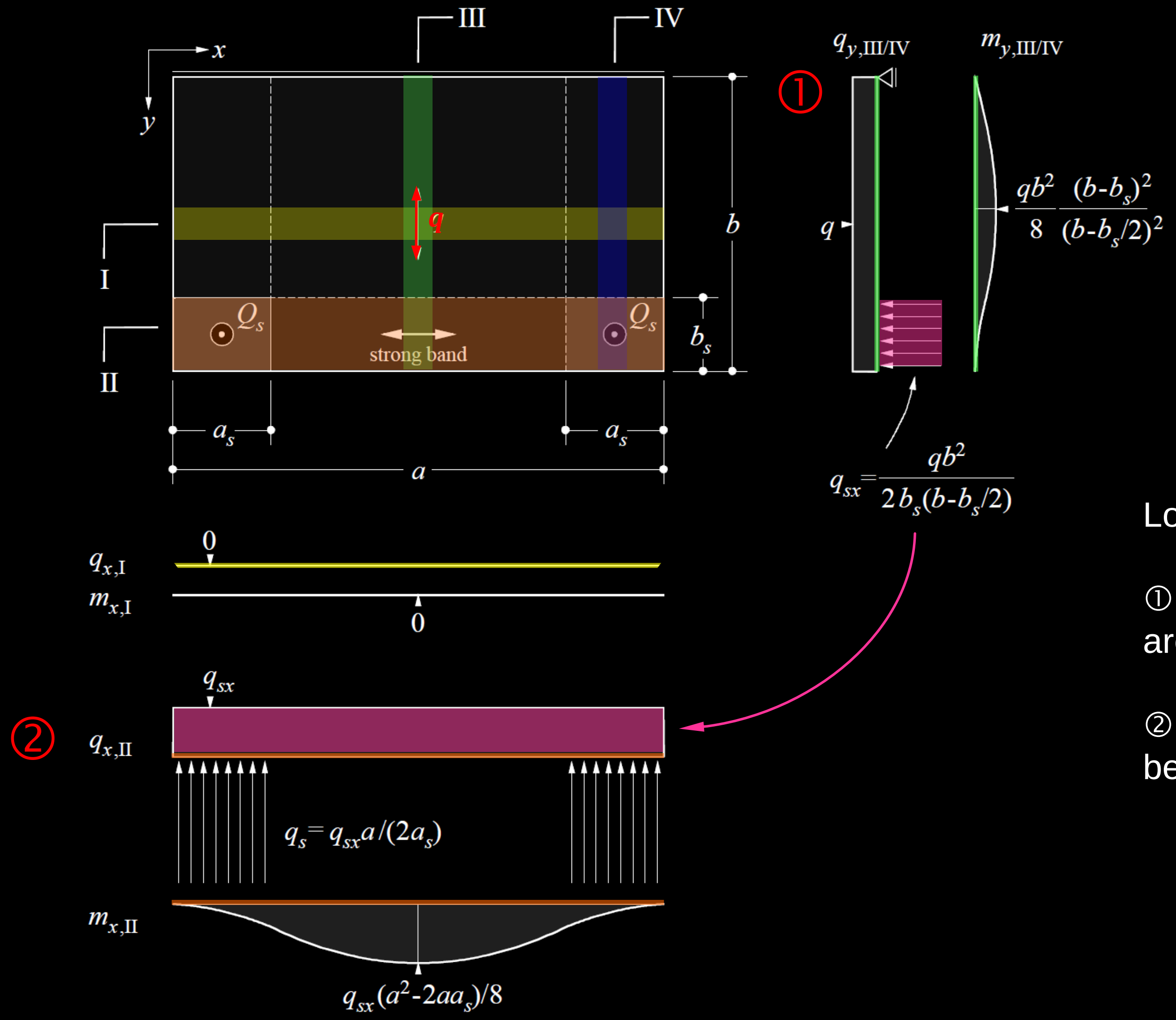
$$m'_{y,u} \geq \frac{Q}{8}\frac{l_y}{l_x} \qquad m_{y,u} \geq 0.034 \cdot q l_y^{\,2} = 0.034 \cdot Q \frac{l_y}{l_x}$$

(complete solution)

(Solutions correspond to upper limit values, but column dimensions are finite and lower limit value $q \geq 4\left(1+\lambda\right) m_u / l^2$ from moment fields is strongly on the safe side → for design ok)

Slabs - Additions

Advanced strip method: Example rectangular slab, simply supported on one side, supported on 2 supports



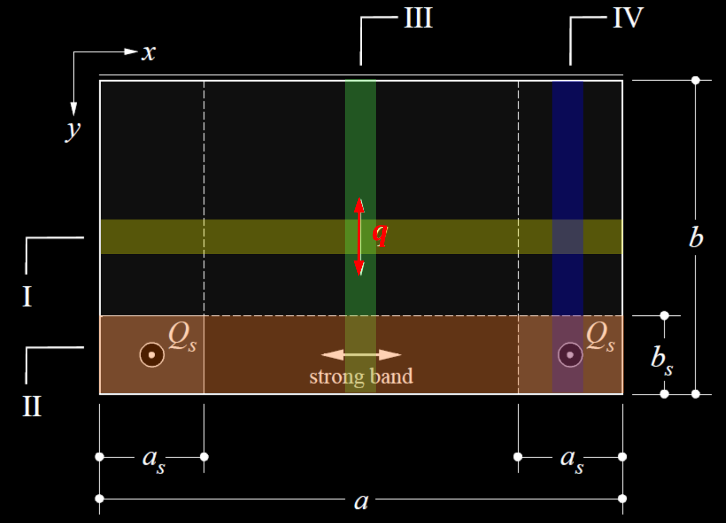
Load transfer:

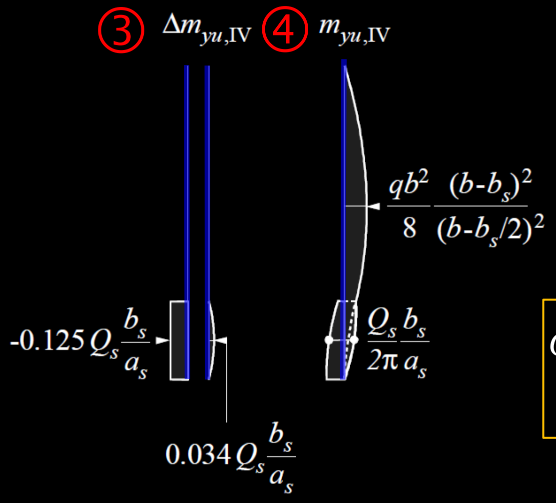
① Total load first carried in *y*-direction (hidden beam as area bearing $a \cdot b_s$)

② Transfer of the reactions on $a \cdot b_s$ by the hidden beam in *x*-direction to the surface bearings $a_s \cdot b_s$

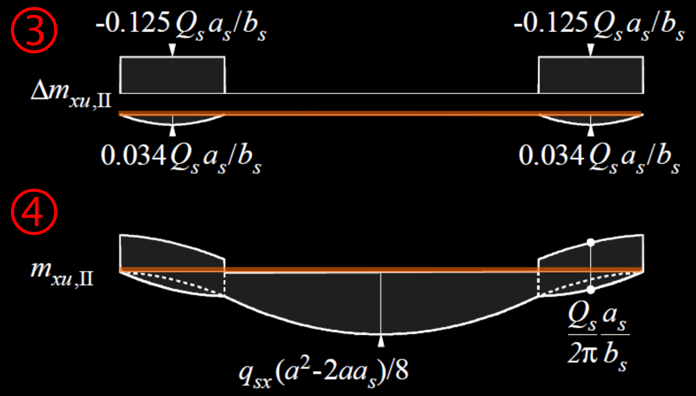
Slabs - Additions

Advanced strip method: Example rectangular slab, simply supported on one side, supported on 2 supports





$$\boxed{Q_s = q_s a_s b_s = \frac{qab}{4}\frac{b}{\left(b - {}^{b_s}\!/_{2}\right)}}$$



Load transfer:

③ Determination of the required bending resistances for the conversion of the area load q_s on $a_s \cdot b_s$ into the single load Q_s

④ Superposition of all necessary bending resistances

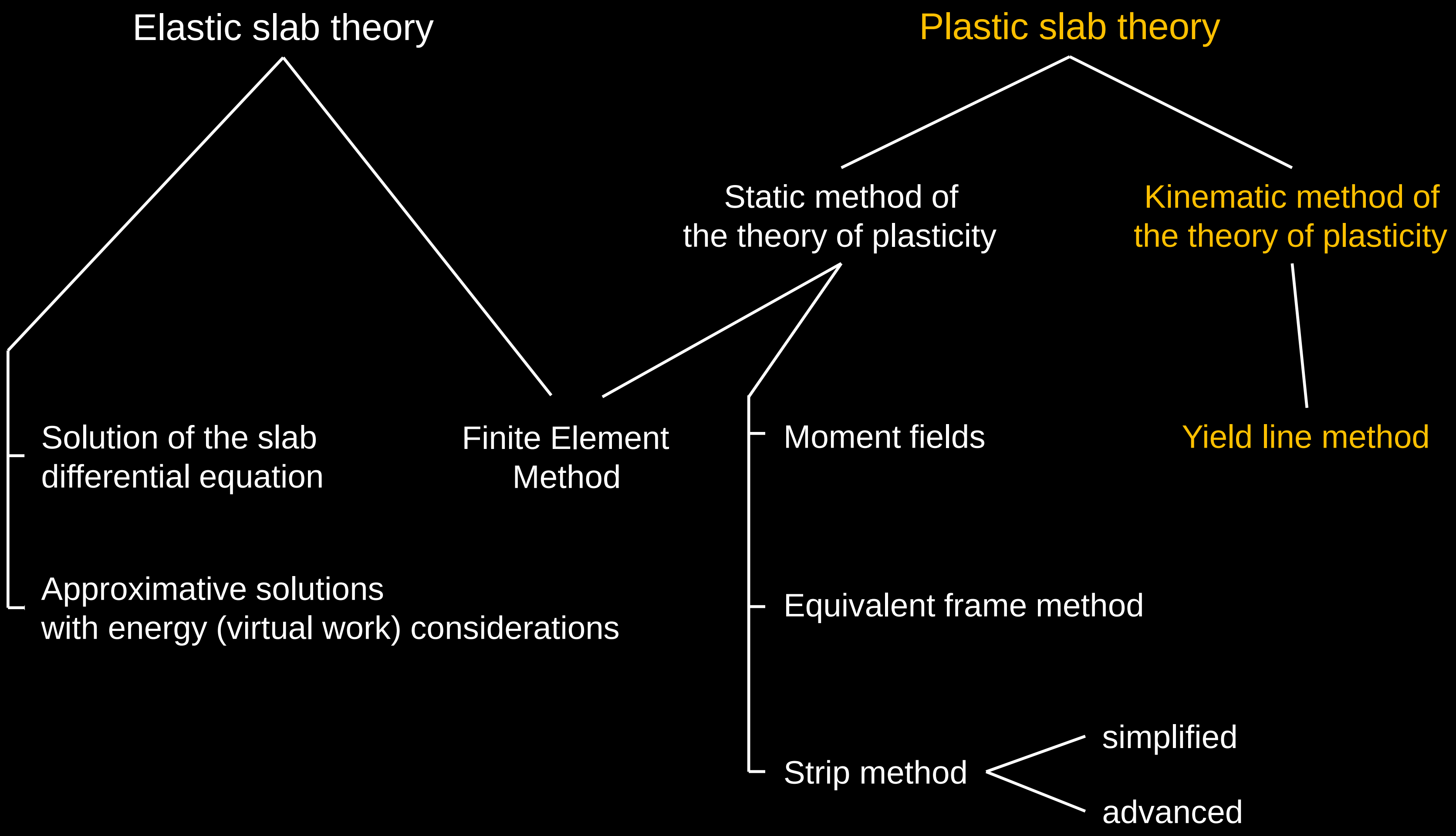
5 Slabs

In-depth study and additions to Stahlbeton II (Chapter 7.3)

5.4 Failure mechanisms

Slabs – Failure mechanisms

Structural analysis / Calculation methods - Overview



Elastic slab theory

Plastic slab theory

Static method of
the theory of plasticity

Kinematic method of
the theory of plasticity

Solution of the slab
differential equation

Finite Element
Method

Moment fields

Yield line method

Approximative solutions
with energy (virtual work) considerations

Equivalent frame method

Strip method

simplified

advanced

Slabs – Failure mechanisms

Yield line method

- The yield line method (Johansen, 1962) is an application of the kinematic method of the theory of plasticity.

- *Procedure*: Assume a kinematically admissible mechanism, then equate the external work done by the applied loads with the internal work (dissipation in rotating yield lines).
  → upper limit value for ultimate load.

- Usually different failure mechanisms have to be investigated, whereby for each mechanism the ultimate load has to be minimised with regard to possible free parameters.

- Rigid parts of the mechanisms usually have a high degree of internal static indeterminacy in contrast to beam structures. A strict plasticity verification (check that the yield conditions are not violated inside the rigid parts) is therefore hardly possible, except in simple special cases.

Slabs – Failure mechanisms

Yield line method

- In comparison to solutions based on the elastic slab theory or equilibrium solutions, the yield line method is quite easy to apply, especially for the verification of existing structures → The kinematic method of the theory of plasticity has become much more widespread for slabs than for beams and membrane elements (very widespread especially in Scandinavia, also for design).

- The "equilibrium method" (Ingerslev, 1923) can be used to circumvent the analytical minimisation process, which is often complex, when using the yield line method. Here, equilibrium is formulated at the individual, rigid slab parts of a mechanism, whereby so-called "nodal forces" are to be considered. However, the method is only valid to a limited extent (partly disproven recently), and the minimisation process can be carried out without any problems using numerical methods today. It is therefore not dealt with in this course.

Slabs – Failure mechanisms

Yield line method – Dissipation (internal work) in a yield line

- Slab, orthogonally reinforced (*x*, *y*)
- Yield line in any direction *t*: Neglecting membrane forces ($n_n = 0$), it applies:

$$dD = m_n \dot{\omega}_n dt$$

- Using the relationship:

$$m_{nu} = m_{xu} \cos^2 \varphi + m_{yu} \sin^2 \varphi$$

- Results in the dissipation:

$$dD = \left(m_{xu} \cos^2 \varphi + m_{yu} \sin^2 \varphi \right) \dot{\omega}_n dt$$

- With rotational velocities around the y- or x-axis:

$$\dot{\omega}_x = \dot{\omega}_n \cos \varphi, \quad \dot{\omega}_y = \dot{\omega}_n \sin \varphi$$

$$d_y = dt \cos \varphi, \quad d_x = dt \sin \varphi$$

→ Dissipation:

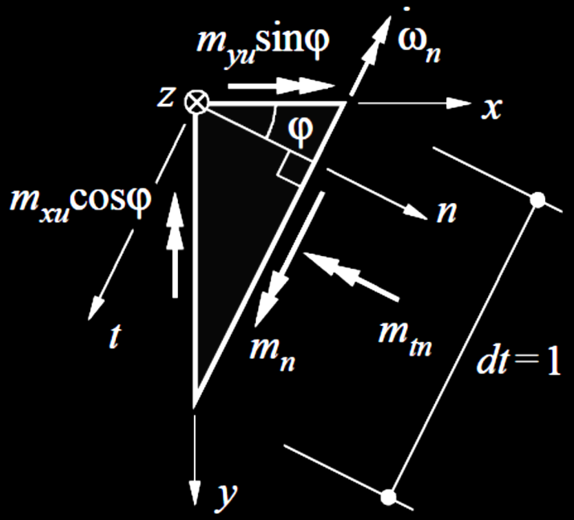
$$\boxed{dD = m_{xu} \dot{\omega}_x dy + m_{yu} \dot{\omega}_y dx}$$

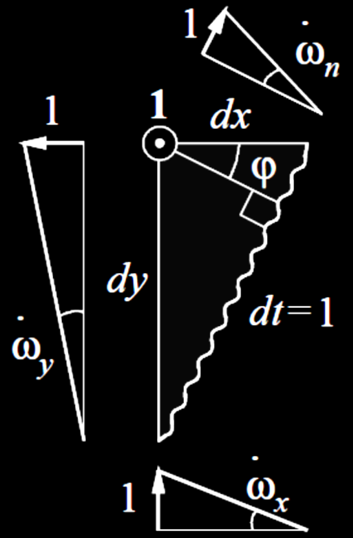
= Sum of the products in the two reinforcement directions of:

bending resistance

rotational velocities around the corresponding axis

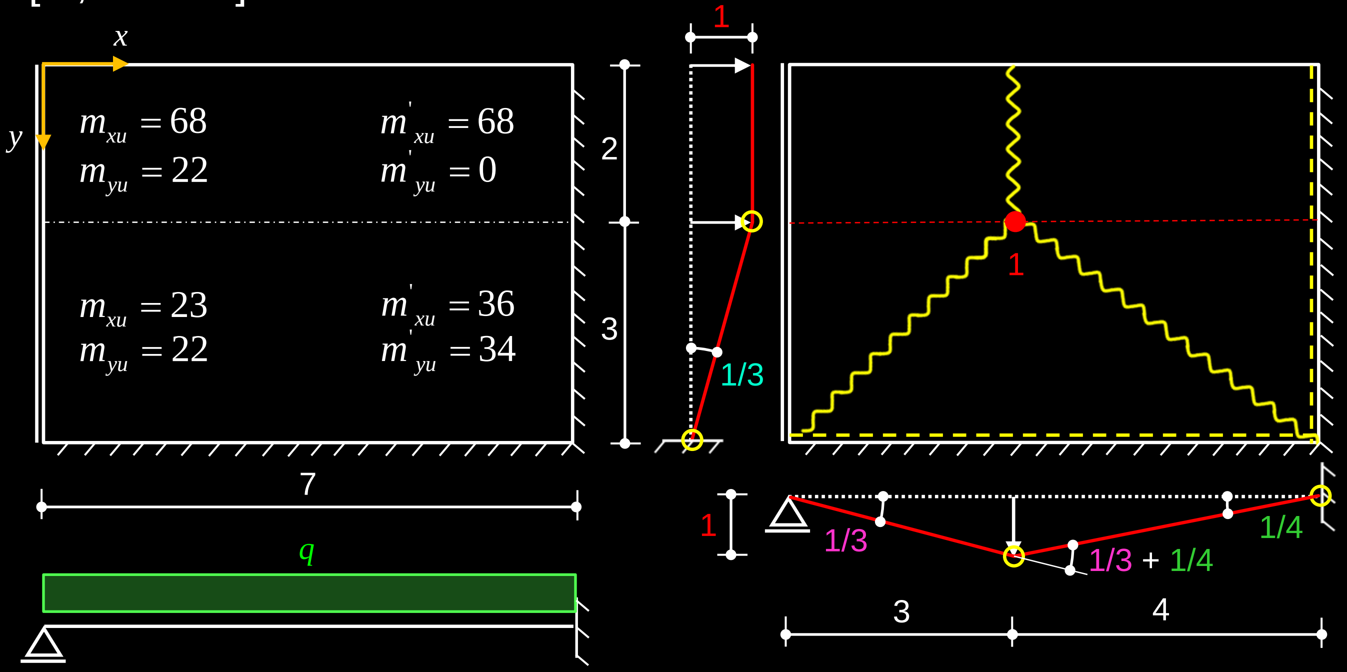
length of the yield line projected onto this axis





Slabs – Failure mechanisms

Example, units [m, kNm/m]

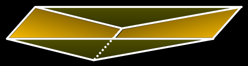


Signatures for yield lines
(n = direction of the normal of the border)

〰 positive yield line,
$m_n = m_{nu}$

- - - - negative yield line,
$m_n = -m_{nu}' = \lambda m_{nu}$

$\lambda = m_u'/m_u$

Work of external forces $W =$ (pyramid  + prism ) - q $= 1 \cdot \left(3 \cdot 7 \cdot \frac{1}{3} + 2 \cdot 7 \cdot \frac{1}{2}\right) q = 14q$

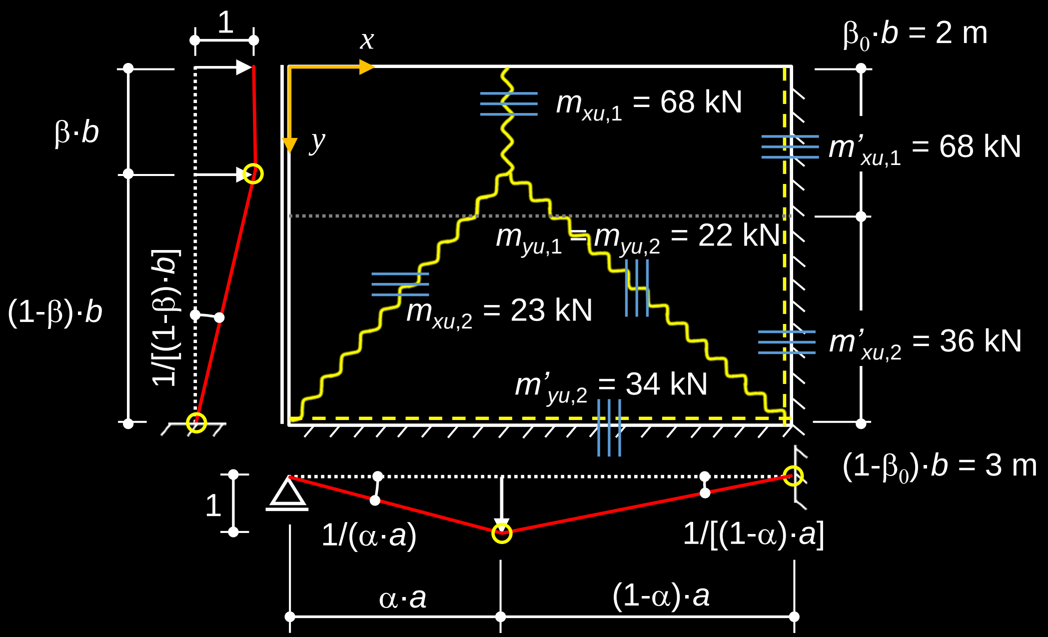
Dissipation work

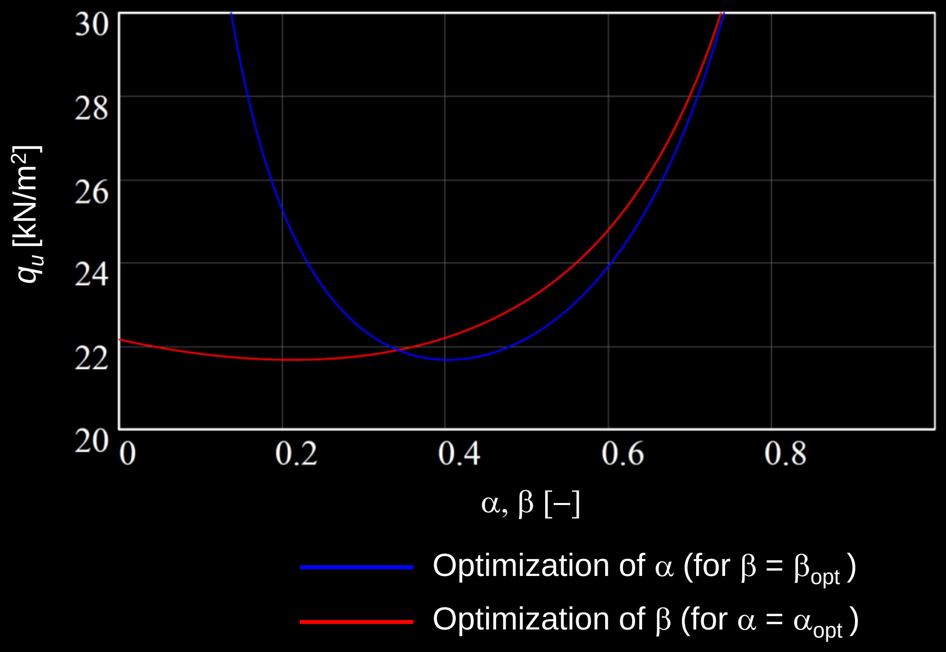
$$D = \underbrace{68 \cdot \left(\frac{1}{3} + \frac{1}{4}\right) \cdot 2 + 68 \cdot \frac{1}{4} \cdot 2 + 23 \cdot \left(\frac{1}{3} + \frac{1}{4}\right) \cdot 3 + 36 \cdot \frac{1}{4} \cdot 3}_{\text{x - dir.}} + \underbrace{22 \cdot \frac{1}{3} \cdot 7 + 34 \cdot \frac{1}{3} \cdot 7}_{\text{y - dir.}} = 311.25$$

$$W = D \rightarrow q_u \leq 22.2 kN / m^2$$

Slabs – Failure mechanisms

Example: Optimisation of yield line geometry





$$D = m_{xu,1}\left(\frac{1}{\alpha a}+\frac{1}{(1-\alpha)\alpha}\right)\beta_0 b + m'_{xu,1}\frac{1}{(1-\alpha)a}\beta_0 b + m_{xu,2}\left(\frac{1}{\alpha a}+\frac{1}{(1-\alpha)a}\right)(1-\beta_0)b + m'_{xu,2}\frac{1}{(1-\alpha)a}(1-\beta_0)b + m_{yu,2}\frac{1}{(1-\beta)b}a + m'_{yu,2}\frac{1}{(1-\beta)}a$$

$$W = \left[(1-\beta)\cdot b\cdot a\cdot\frac{1}{3}+\beta\cdot b\cdot a\cdot\frac{1}{2}\right]\cdot q = a\cdot b\cdot\frac{\beta+2}{6}\cdot q \quad \rightarrow \quad \frac{\partial q_u}{\partial\alpha}=0 \text{ and } \frac{\partial q_u}{\partial\beta}=0 \rightarrow \quad q_{u,opt}(\alpha\cdot a = 2.823\text{ m}; \beta\cdot b = 1.062\ m) = 21.7\text{ kPa}$$

Conclusion: Despite the strong differences in geometry, the ultimate load deviates only slightly (flat minima)!

Slabs – Failure mechanisms

Yield line method - Fan mechanisms

- Slab, isotropically reinforced ($m_{xu} = m_{yu} = m_u$)

- Principal radius of curvature in cone element

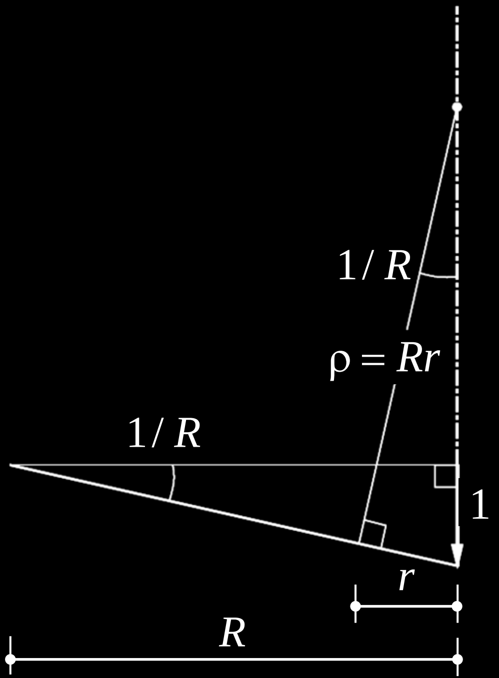
from $\rho = Rr \qquad \frac{r}{\rho} = \frac{1}{R}$

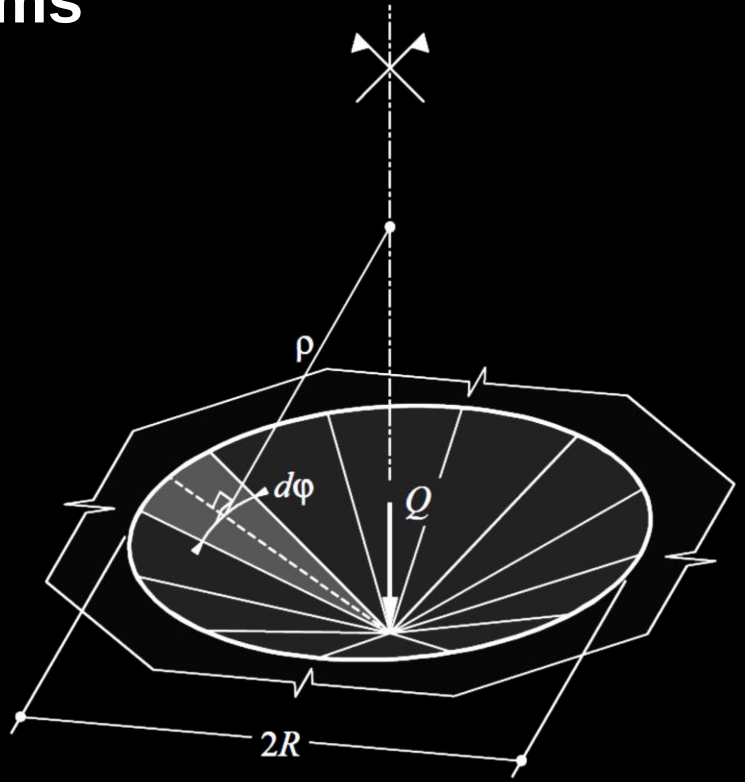
→ Principal curvature $\chi_1 = \rho^{-1} = (Rr)^{-1}$

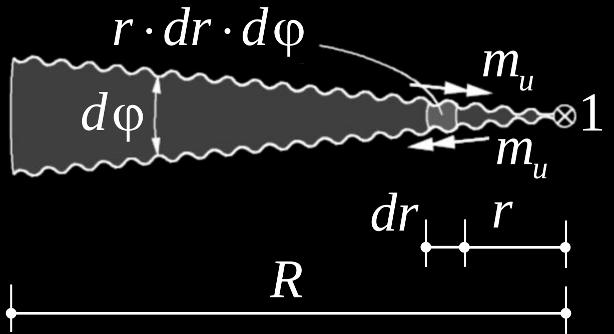
→ Rotation $\dot{\omega}_\varphi = \chi_1 r d\varphi$

- Dissipation per area element in the fan:

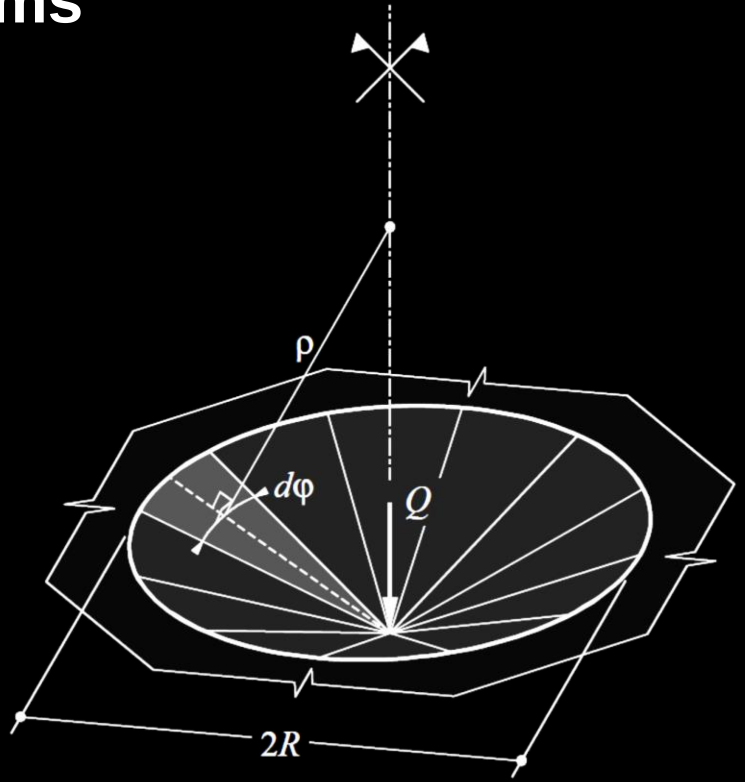
$$dD = m_u \dot{\omega}_\varphi dr = m_u \frac{1}{\rho} r d\varphi dr$$







Slabs – Failure mechanisms



Yield line method - Fan mechanisms

- Dissipation per area element in the fan:

$$dD = m_u \dot{\omega}_\varphi dr = m_u \frac{1}{\rho} r d\varphi dr$$

Dissipation inside a fan with opening angle β:

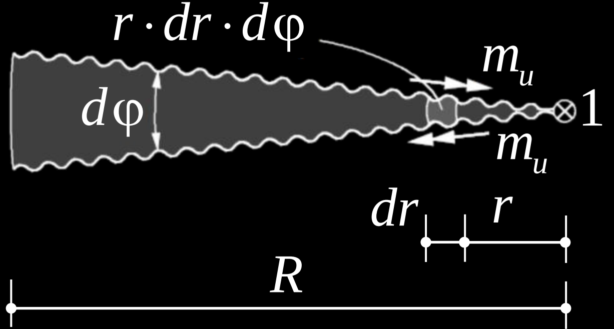
$$D = \left\{ \int_0^\beta \frac{1}{R(\varphi)} \int_0^{R(\varphi)} m_u(r,\varphi) dr \right\} d\varphi \qquad \text{with} \qquad \rho = Rr$$

- where m_u and R can be general functions of angle φ
- Dissipation along the fan boundary (independent of R):

$$D = \int_0^\beta \frac{1}{R} m'_u \, R d\varphi = \int_0^\beta m'_u(r,\varphi) d\varphi$$

→ Dissipation in a fan with opening angle β for constant m_u and $m'_u = \lambda m_u$:

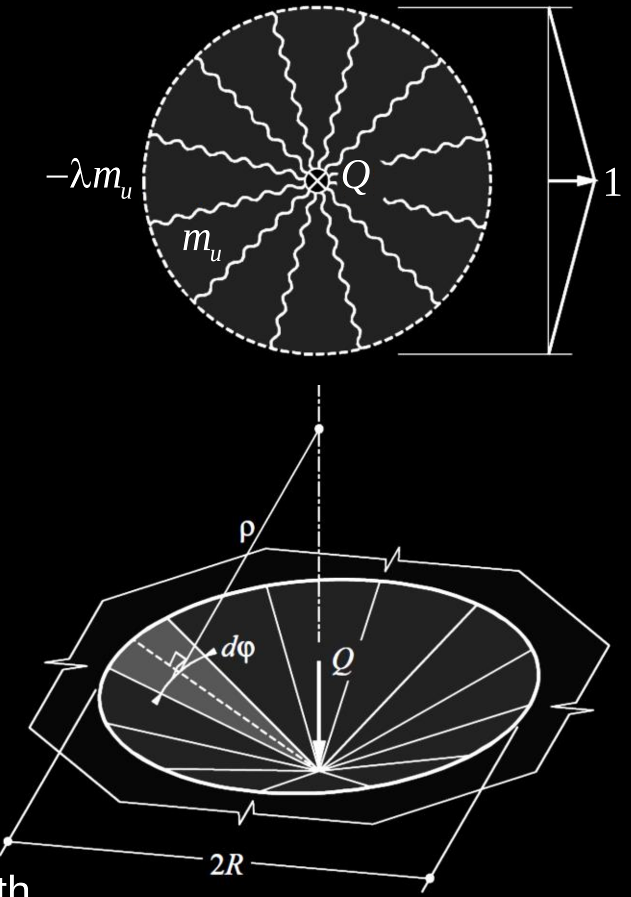
$$D = \beta(m_u + m'_u) = \beta m_u (1 + \lambda)$$



Slabs – Failure mechanisms

Concentrated load on slab of any geometry

$$\left.\begin{aligned} W &= Q\cdot 1 \\ D &= 2\pi m_u (1+\lambda) \end{aligned}\right\} \quad \boxed{Q_u \leq 2\pi\left(m_u + m'_u\right) = 2\pi m_u\left(1+\lambda\right)}$$



Same ultimate load as with moment field for a centrically supported circular slab under uniform load (independent of *R*) → complete solution for a circular slab. Upper limit value for other cases.

By applying the transformation theorem (*), the upper limit value is obtained for an orthotropically reinforced slab of any geometry:

$$\boxed{Q_u \leq 2\pi\left(\sqrt{m_{xu} m_{yu}} + \sqrt{m'_{xu}\, m'_{yu}}\right) = 2\pi\sqrt{m_{xu} m_{yu}}\,(1+\lambda)}$$

(*) A valid solution for a slab isotropically reinforced with bending resistances m_u, m'_u under loads *q* and *Q* in the coordinates (*x*,*y*), can be applied to an orthotropically reinforced slab with $m_{yu} = \mu\cdot m_{xu} = \mu\cdot m_u$, $m'_{yu} = \mu\cdot m'_{xu} = \mu\cdot m'_u$. The coordinates are to be transformed with $x^* = x$, $y^* = y\sqrt{\mu}$, the loads with $q^* = q$ and $Q^* = Q\sqrt{\mu}$

(The practical use is limited. For example, an isotropically reinforced square slab corresponds to an orthotropically reinforced slab with stronger reinforcement in the longer direction, which is unpractical).

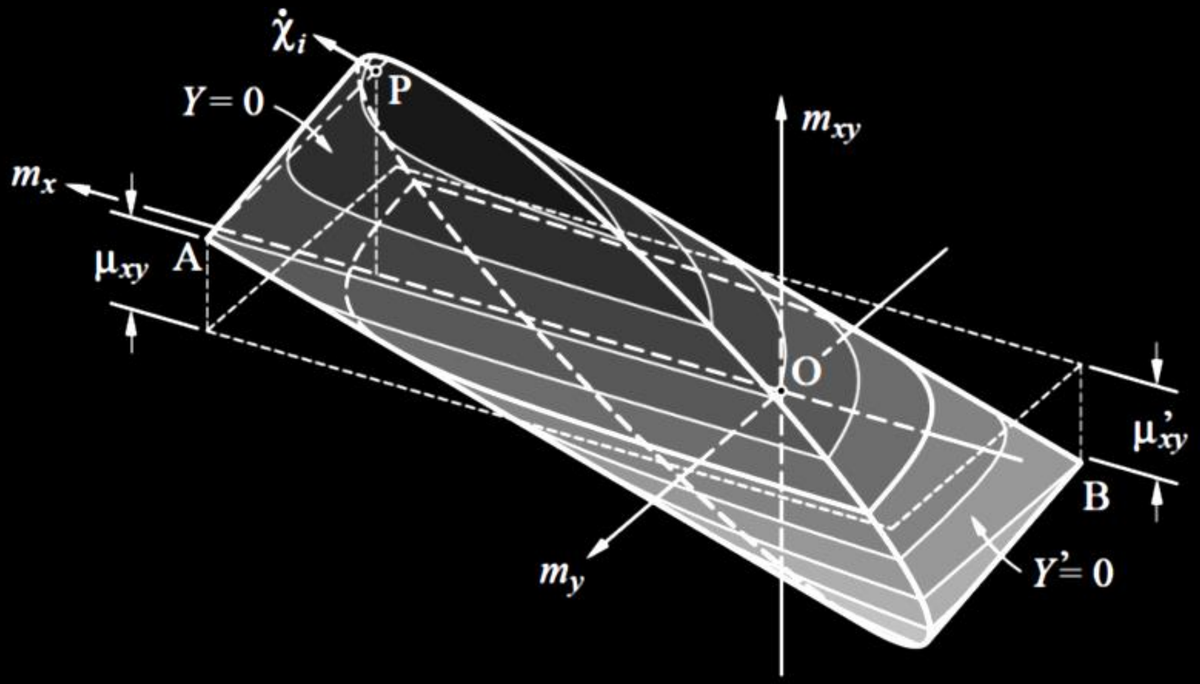
5 Slabs

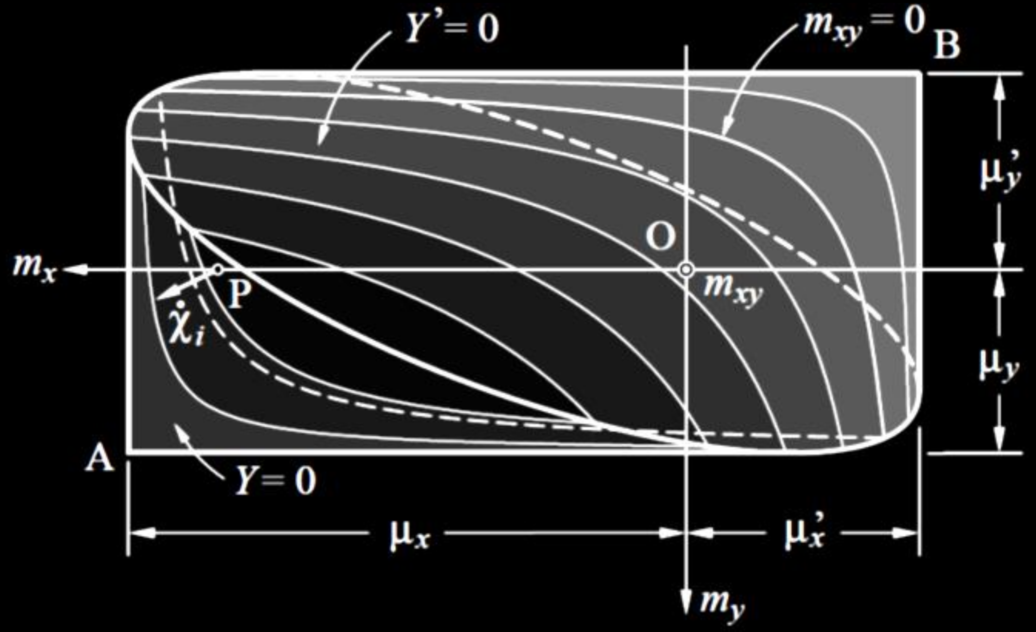
Appendix 1

Slabs - Yield conditions

Yield conditions for skew reinforcement

Representation of the yield condition:
(two elliptical cones; compare with orthogonal reinforcement where the peaks lie in the plane $m_{xy} = 0$ and the intersecting ellipse in a plane parallel to the m_{xy}-*axis*).





[Seelhofer (2009)]

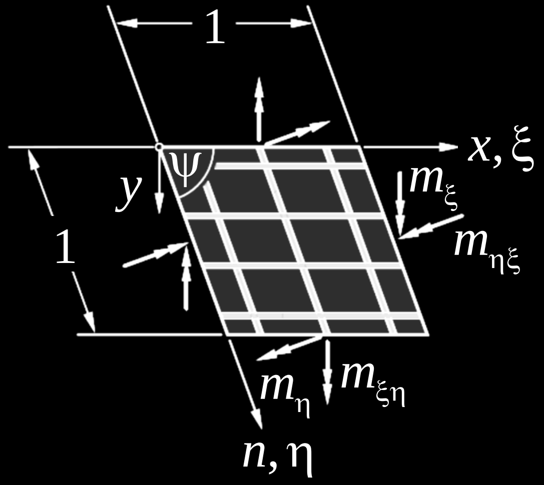
Slabs - Yield conditions

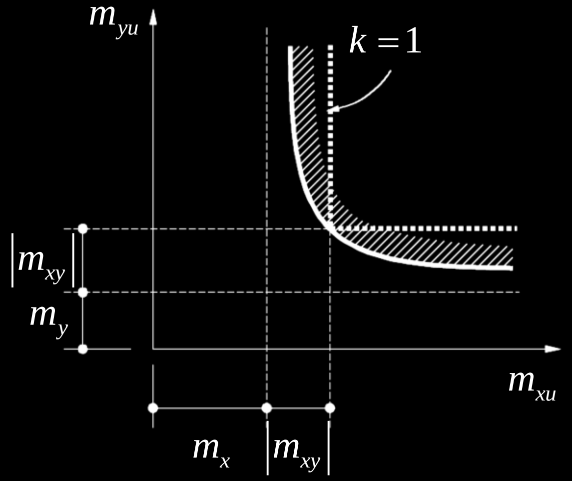
Skew reinforcement

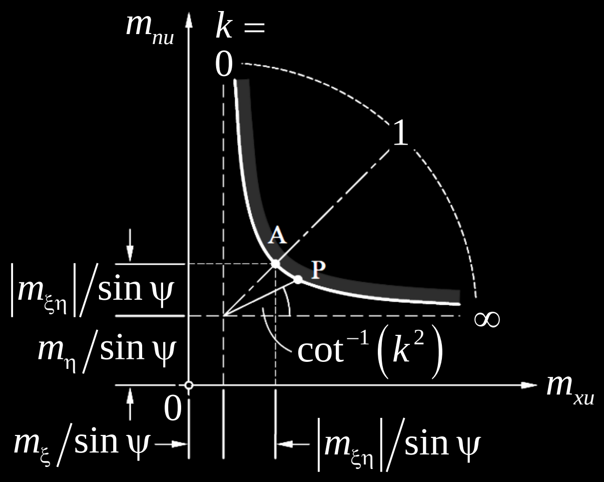
Using the parametric form, the design (and the graphical representation) is possible analogous to orthogonal reinforcement.

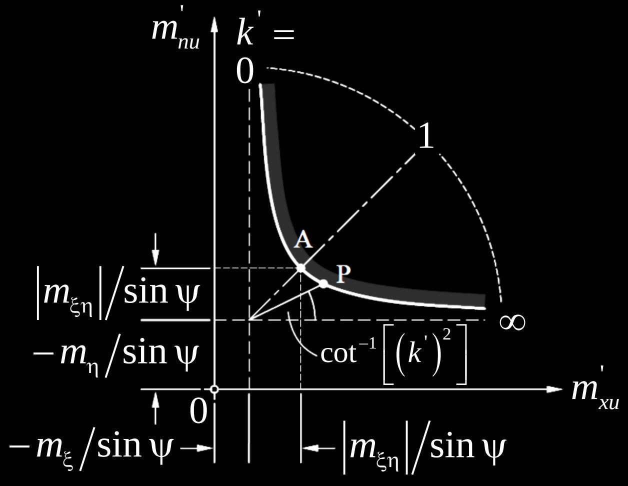
$$m_{xu} \geq \frac{1}{\sin\psi}\left(m_\xi + k\left|m_{\xi\eta}\right|\right) \qquad m_{nu} \geq \frac{1}{\sin\psi}\left(m_\eta + k^{-1}\left|m_{\xi\eta}\right|\right) \qquad k = \left|\sin\psi\tan\varphi_u + \cos\psi\right|$$

$$m_{xu}^{'} \geq \frac{1}{\sin\psi}\left(-m_\xi + k^{'}\left|m_{\xi\eta}\right|\right) \qquad m_{nu}^{'} \geq \frac{1}{\sin\psi}\left(-m_\eta + \left(k^{'}\right)^{-1}\left|m_{\xi\eta}\right|\right) \qquad k^{'} = \left|\sin\psi\tan\varphi_u^{'} + \cos\psi\right|$$









If no upper or lower reinforcement is required in one of the two reinforcement directions, refer to Seelhofer (2009).