4 Long-term effects

Specialisations and additions to Stahlbeton I

4.1 Basics

Learning objectives

Within this chapter, the students are able to:

- understand the processes and influencing parameters of shrinkage, creep, and relaxation and draw the diagrams of stress/strain against time for all processes.
- discuss the effect of creep on the deformations and internal forces in statically determinate and indeterminate structures.
- elaborate on the long-term behaviour due to creep of structures subjected to (i) fast or slow settlements and (ii) changes in the structural system (comparing e.g. one cast systems and staged construction).
- estimate with the time-dependent force method the long-term behaviour of a structural system composed of 1D elements with non-uniform creep properties.

Time-dependent behaviour of concrete

Shrinkage

Volume contraction without load

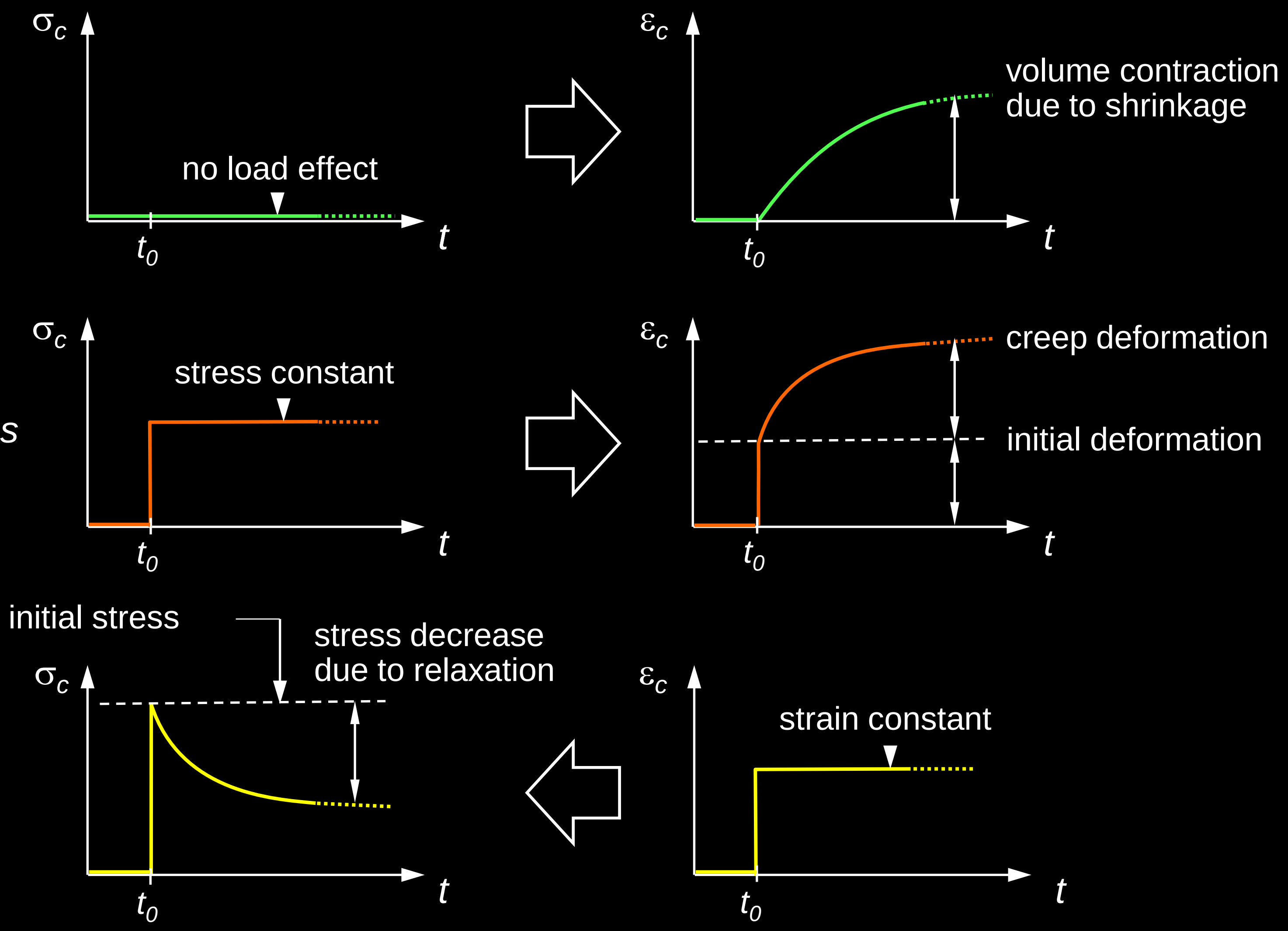
(Figure for free, unrestrained deformations
→ no restraint forces)

Creep

Increase of deformations under constant stress

Relaxation

Decrease of stress under constant strain



Long-term effects

Shrinkage

Early/capillary (Plastic) shrinkage (up to 4‰ → avoid!)

- Capillary stresses during the evaporation of water from the fresh concrete lead to a denser structure of the cement matrix in the first few hours until hardening.
- Avoidance through careful curing (prevention of significant water losses on the fresh concrete surface caused by high concrete or air temperatures, low humidity, and wind).

Autogenous and chemical shrinkage (normal concrete up to 0.3‰, UHPC up to 1.2‰)

- Volume contraction during hydration, initially caused by the chemical integration of the water molecules into the hydration products (first days). Afterwards, as soon as the water in the capillary pores is used up, it is mainly caused by capillary tension due to the lower internal relative humidity, leading the hydration to consume water from the gel pores (first weeks).
- Primarily dependent on W/C ratio: The lower the W/C ratio, the greater the autogenous shrinkage (significant effect only for W/C < 0.45 high-performance concrete, UHPC).

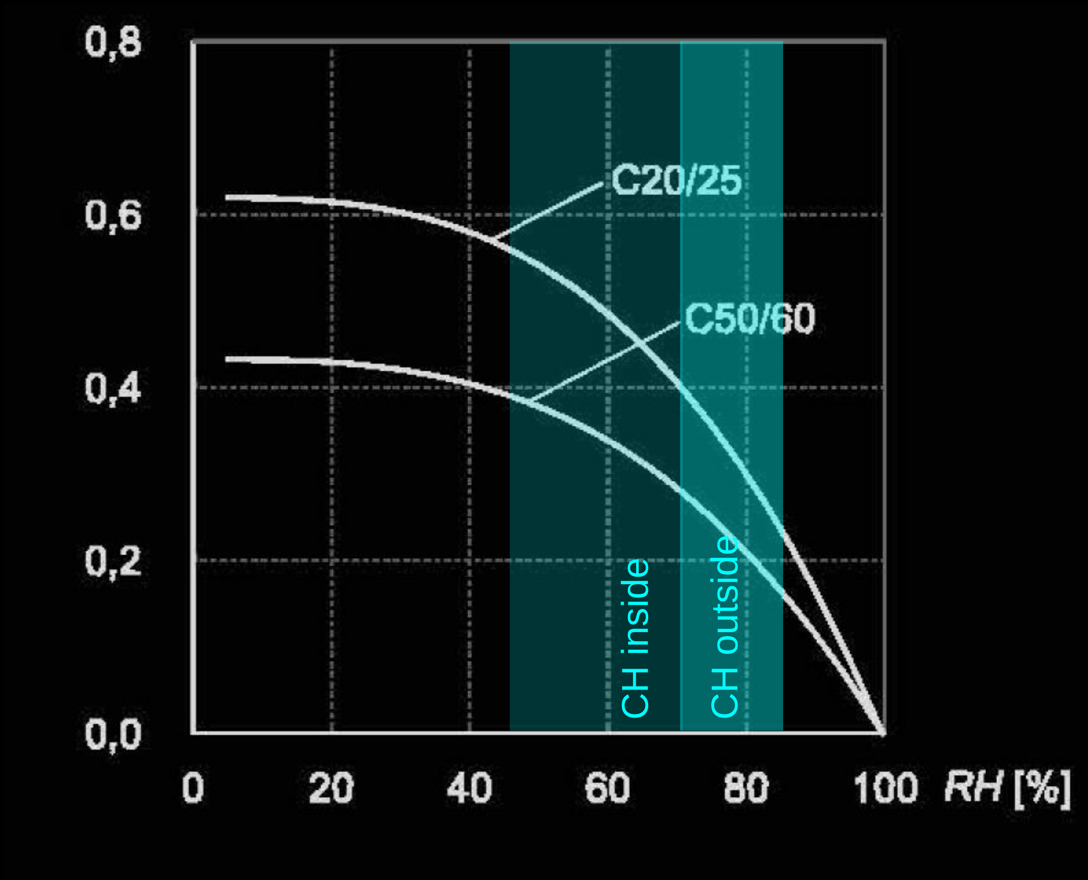
Drying shrinkage (up to approx. 0.3‰ outside at RH=70%, up to approx. 0.5‰ inside at RH=50%)

- Volume contraction in hardened concrete by releasing water into the environment. Begins with formwork stripping or the end of curing and lasts for years.
- Magnitude primarily dependent on cement paste volume (cement, admixtures, entrapped air, and water). Faster for high W/C ratios, low air humidity, and thin components.

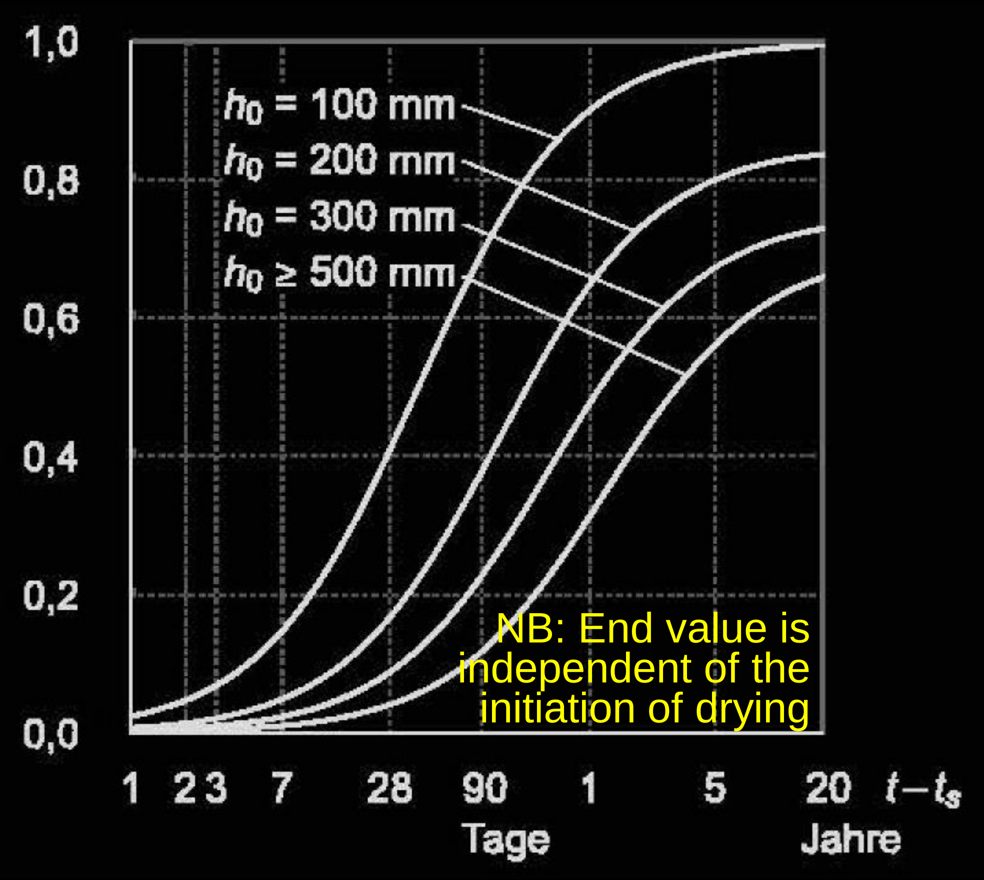
Time-dependent behaviour of concrete

Drying shrinkage ε_{cd}
(according to SIA 262)

Drying shrinkage $\varepsilon_{cd\infty}$ [‰]



Progression $\varepsilon_{cd}(t)/\varepsilon_{cd\infty}$



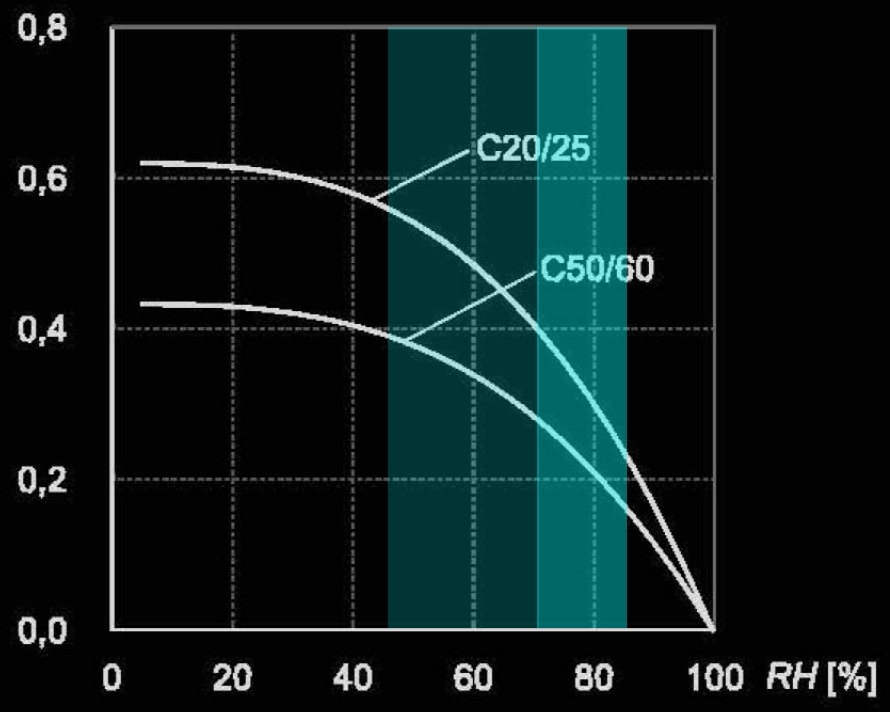
Time-dependent behaviour of concrete

Drying shrinkage ε_{cd}

Autogenous shrinkage ε_{ca} (according to SIA 262)

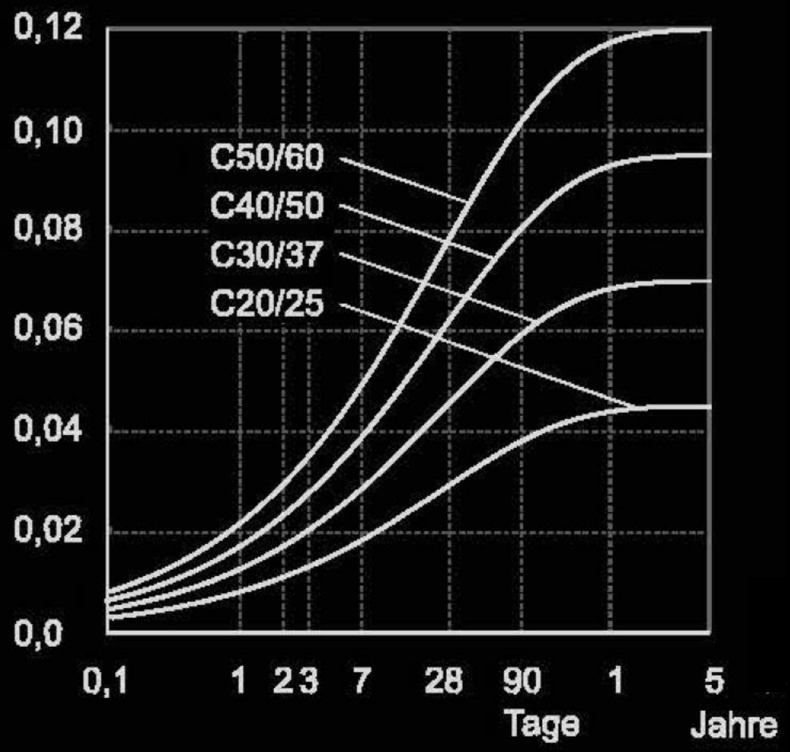
$$\varepsilon_{cs}(t) = \varepsilon_{cd}(t) + \varepsilon_{ca}(t)$$

Final drying shrinkage value $\varepsilon_{cd\infty}$ [‰]



+

Progression of autogenous shrinkage $\varepsilon_{ca}(t)$ [‰]



Time-dependent behaviour of concrete

Drying shrinkage ε_{cd}

Autogenous shrinkage ε_{ca} (according to SIA 262)

$$\varepsilon_{cs}(t) = \varepsilon_{cd}(t) + \varepsilon_{ca}(t)$$

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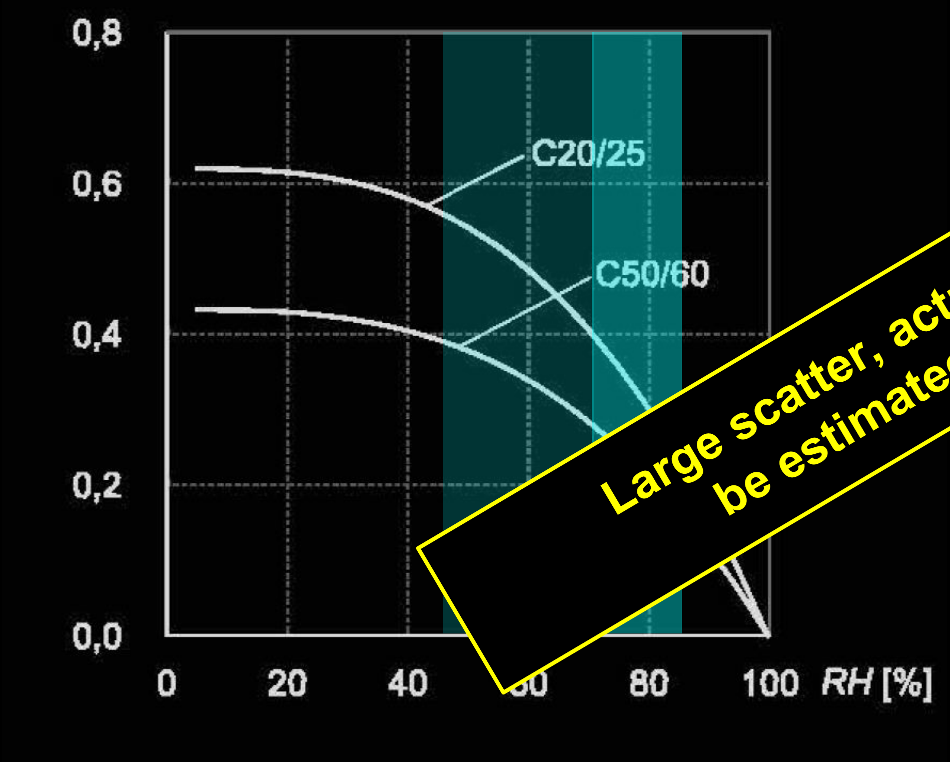
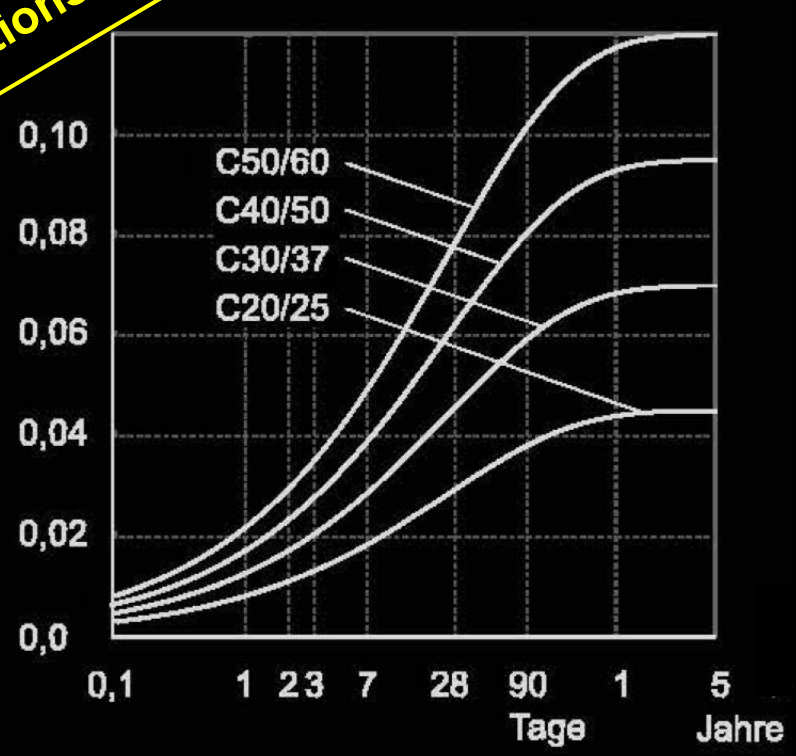


Large scatter, actual shrinkage deformations can only be estimated even with complex calculations!

autogenous shrinkage $\varepsilon_{ca}(t)$ [‰]



Long-term effects

Creep and relaxation

Cause / Phenomena

- Stress leads to rearrangement or evaporation of water in the cement paste. The associated sliding and compaction processes lead to volume contraction.
- The standards assume that creep deformations cease after some decades (asymptotically approaching the long-term creep coefficient φ_∞). This is controversial today, there are e.g. older cantilever bridges indicating that creep deformations might keep increasing continuously. Yet, only few experiments are available.

Influences on the magnitude of creep deformations

- Load level (creep deformations approximately proportional to the load)
- Cement paste volume (high cement paste volume = larger creep deformations)
- Concrete compressive strength (high compressive strength = smaller creep deformations)
- Age of the concrete (loading at a young age causes larger creep deformations)

Influences on the course of time

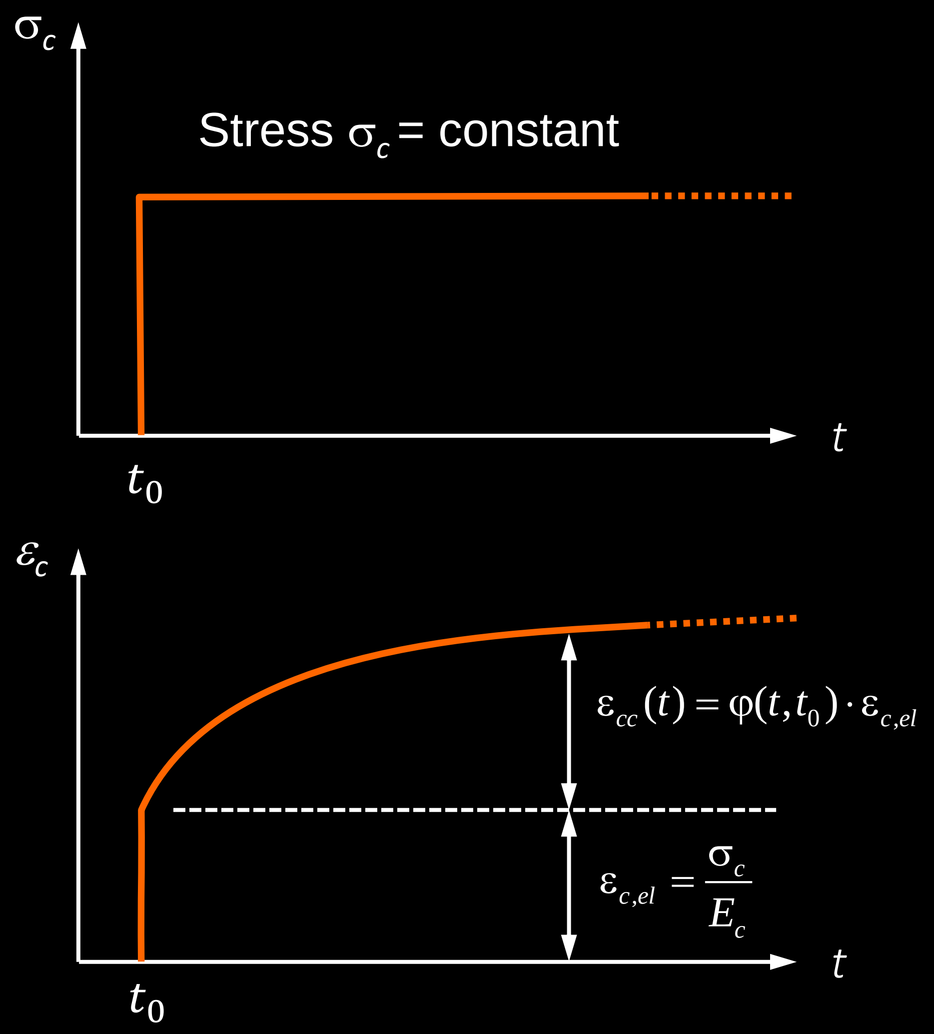
- Creep is faster in smaller elements (thin components)
- Creep is faster at low relative humidity (dry environment)

Relaxation

- Creep and relaxation are related phenomena
- Relaxation behaviour is influenced by the same variables as creep

Long-term effects

Creep



- Increase in deformation under constant stress
- $$\varepsilon_c(t) = \varepsilon_{c,el} + \varphi(t,t_0)\cdot\varepsilon_{c,el} = \left(1+\varphi(t,t_0)\right)\cdot\varepsilon_{c,el}$$

  with

$\varphi(t,t_0)$	creep coefficient
t	time
t_0	age of the concrete at the start of exposure
$t-t_0$	load duration

- Normal case: $\varphi_{t=\infty} \approx 1.5 \ldots 2.5$, i.e. increase of deformations by a factor of 2.5...3.5
- Analogous behaviour under tension (in uncracked concrete)

Long-term effects

Relaxation (≈ creep at ε = const.)

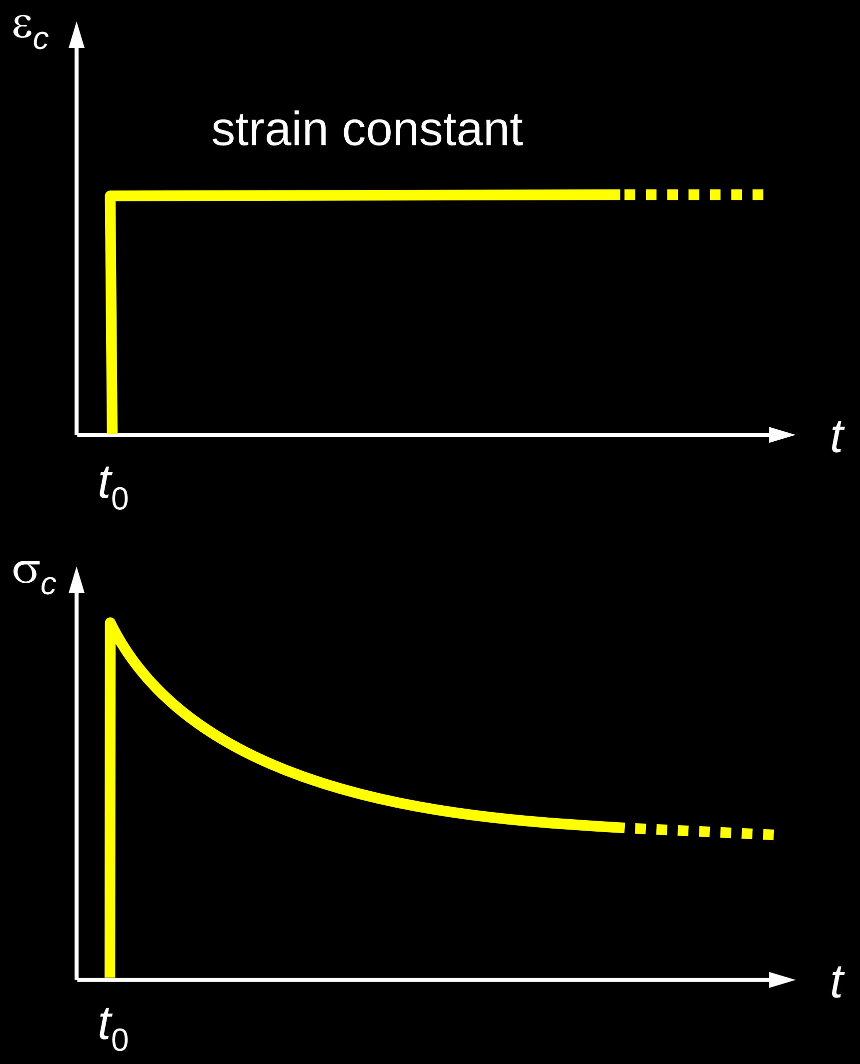
- Decrease in stress under constant strain
- Approximation (fictitious modulus of elasticity):

$$\sigma_c(t) = \sigma_{c,t=0} \cdot \frac{1}{1+\varphi}$$

- Better approximation (derived e.g. using Trost method)

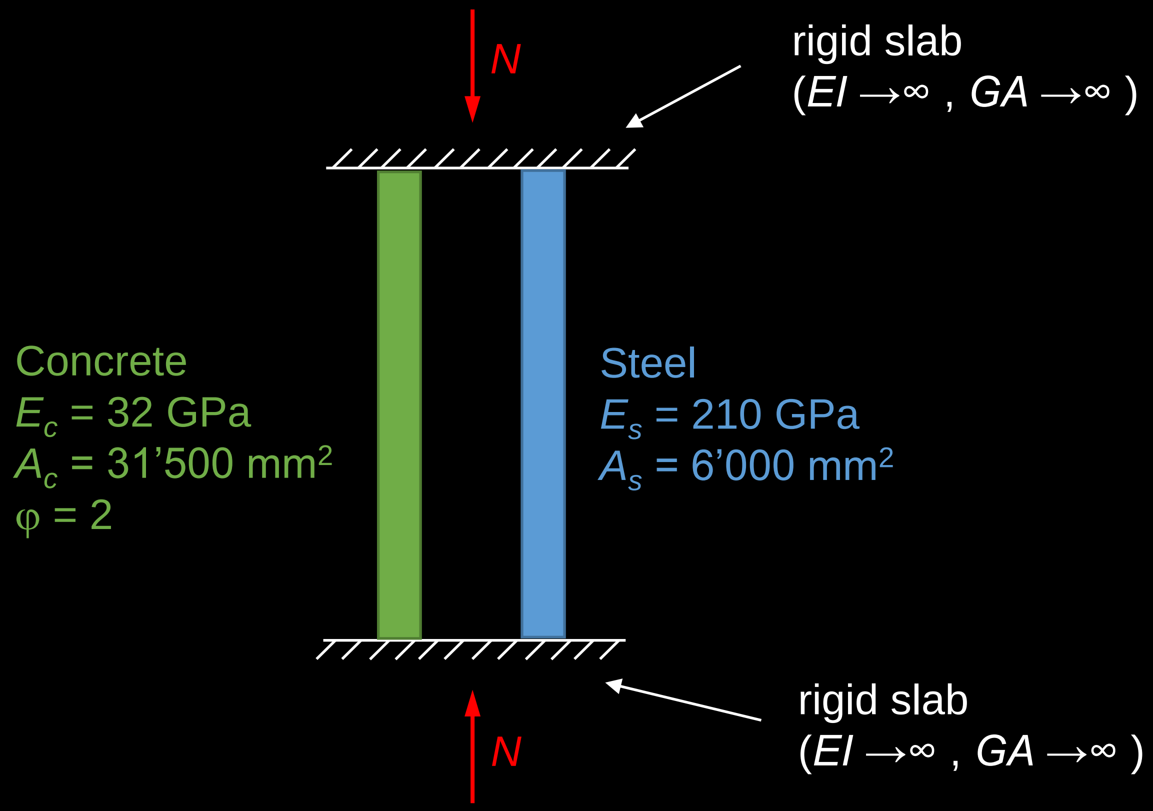
$$\sigma_c(t) = \sigma_{c,t=0} \cdot \left(1 - \frac{\varphi(t)}{1+\mu \cdot \varphi(t)}\right)$$

- Normal case: $\varphi_{t=\infty} \approx 1.5 \ldots 2.5$, μ = ca. 0.75, i.e. reduction of the initial stress to approx. 25%.
- Decrease is less pronounced (to approx. 40%) if deformation is imposed slowly



Long-term effects

In-class exercise



How is the load N distributed between the two columns?

- at $t = 0$
- at $t \rightarrow \infty$

Long-term effects

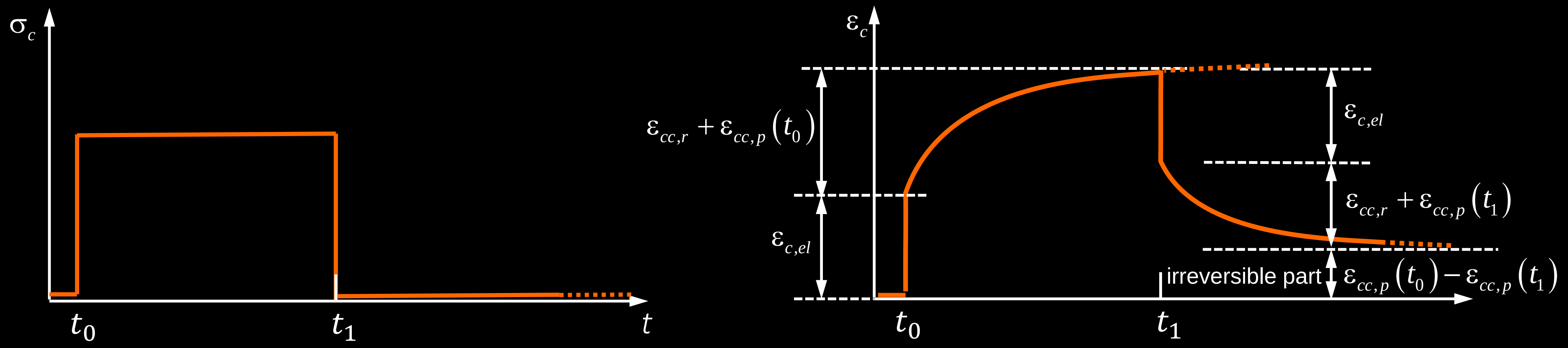
Creep - reversible and plastic part

- The deformations of the concrete under load are composed of the elastic deformations $\varepsilon_{c,el}$ and the time-dependent creep deformations ε_{cc}
- The creep deformations ε_{cc} consist of a reversible component $\varepsilon_{cc,r}$ (reversal sets in relatively quickly, half-life approx. 30 days) and an irreversible (plastic) component $\varepsilon_{cc,p}$:

$$\varepsilon_c(t) = \varepsilon_{c,el} + \varepsilon_{cc,r}(t) + \varepsilon_{cc,p}(t) = \varepsilon_{c,el} + \varepsilon_{cc}(t)$$

The irreversible part $\varepsilon_{cc,p}$ depends on the time of loading = concrete age at load application (old concrete is less prone to creep) and occurs much slower than the reversible component.

- Example: Loading and complete unloading after a longer period of time (permanent stretching):



For simplicity, no distinction is made between the two components in the following slides, and no unloading is considered.

Long-term effects

Creep – Final value and progression
(see also SIA 262, 3.1.2.6)

$$\varphi(t,t_0) = \varphi_{RH} \cdot \beta_{\sigma c} \cdot \beta_{fc} \cdot \beta(t_0) \cdot \beta(t-t_0) \qquad (\approx 1.5 \ldots 2.5)$$

t: time at which the creep coefficient φ is determined

t_0: concrete age at time of loading

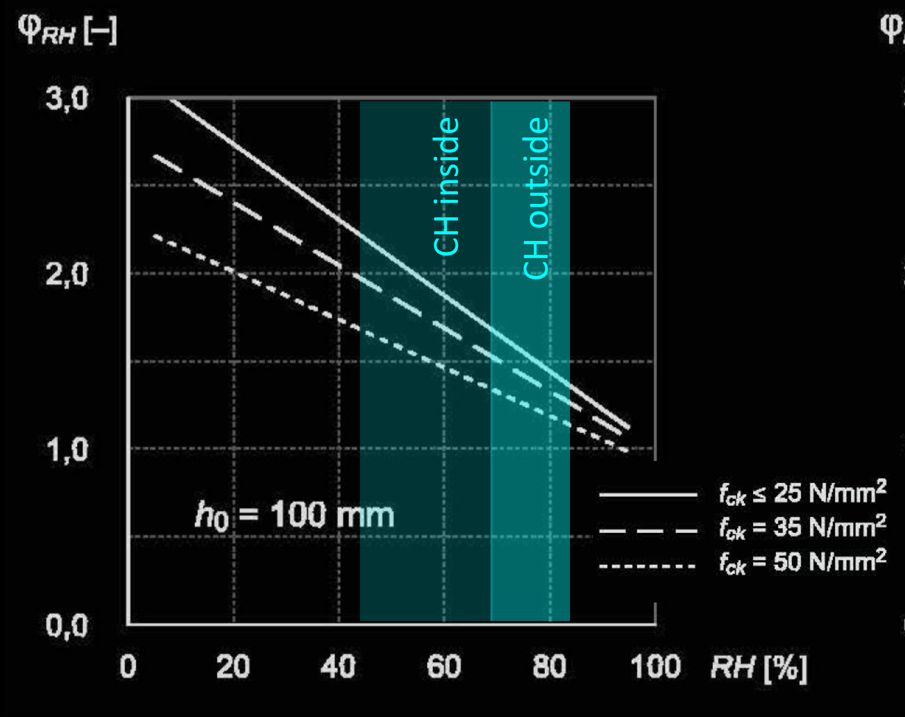
- Relative humidity: φ_{RH} (CH) $\approx 1.25 \ldots 1.5 \;\; (RH \approx 80 \ldots 65\%)$
- Stress level: $\beta_{\sigma c} = e^{1.5\left(\frac{\sigma_c}{f_{ck}} - 0.45\right)}$ (für $\sigma_c > 0.45 f_{ck}$, sonst $\beta_{\sigma c} = 1$)
- Concrete compressive strength: $\beta_{fc} = \begin{matrix} \ldots & C25/30 & C30/37 & C35/45 & \ldots \\ \ldots & 2.9 & 2.7 & 2.6 & \ldots \end{matrix}$
- Concrete age at loading: $\beta(t_0) \approx 1.2 \ldots 0.2 \qquad \beta(t_0 = 28\,\text{d}) = 0.5$
  (corrected for the influence of the temperature: $t_{0,\text{eff}} \rightarrow k_T t_0$)
- Load duration ($\rightarrow$ progression): $\beta\left((t=\infty) - t_0\right) \approx 1$

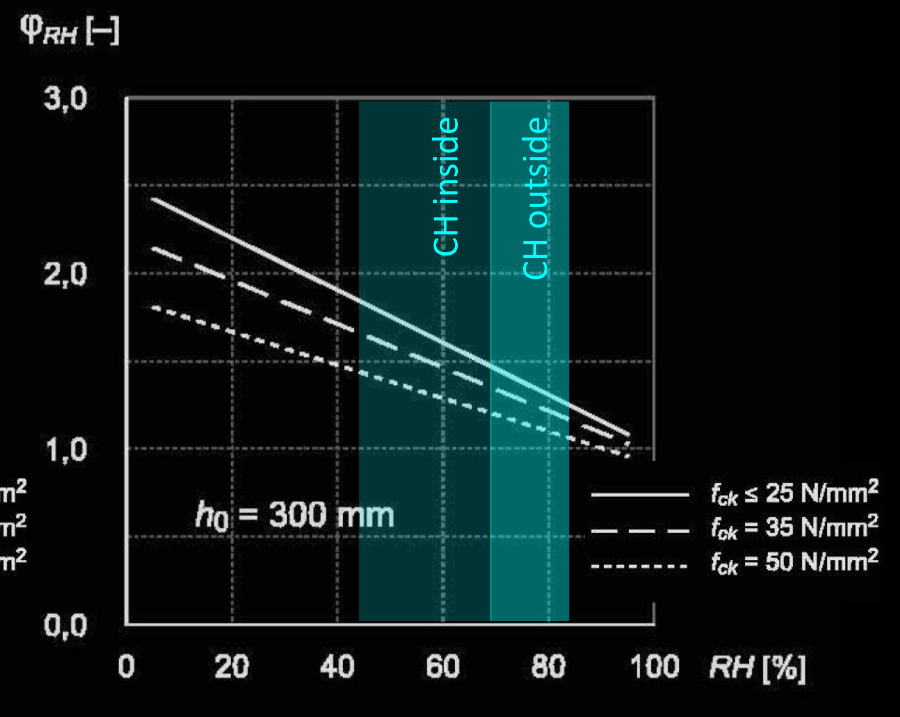
Long-term effects

Creep – Final value and progression
(see also SIA 262, 3.1.2.6)

$$\varphi(t,t_0) = \varphi_{RH} \cdot \beta_{\sigma c} \cdot \beta_{fc} \cdot \beta(t_0) \cdot \beta(t-t_0) \qquad (\approx 1.5 \ldots 2.5)$$

φ_{RH}: Coefficient for relative humidity (RH: usually the annual mean)



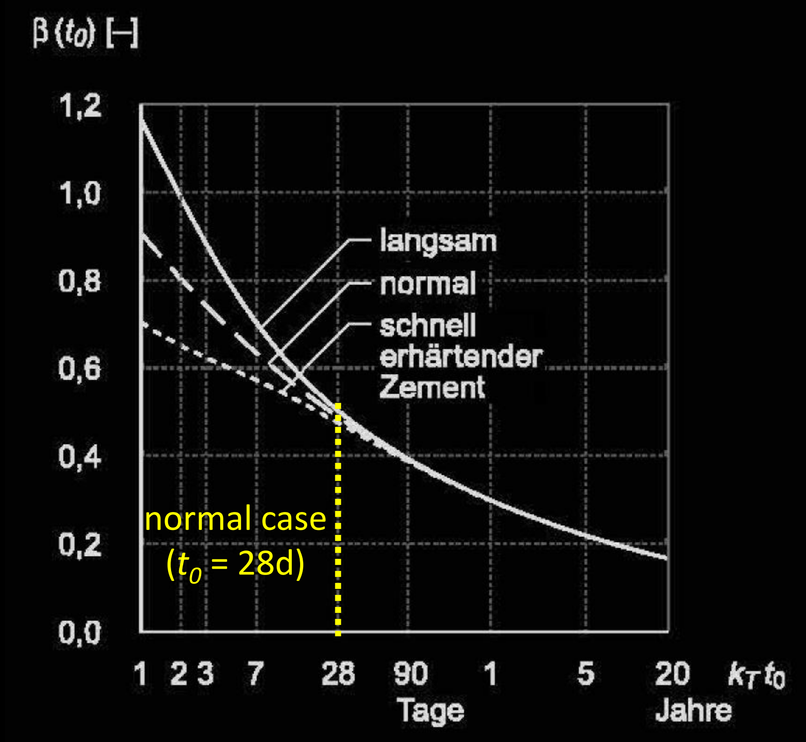


Long-term effects

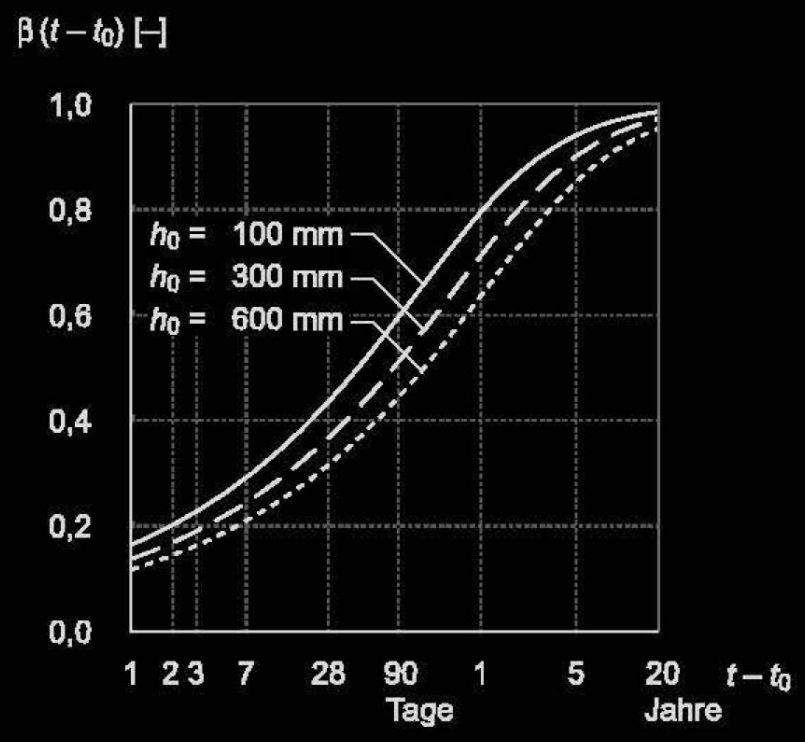
Creep – Final value and progression
(see also SIA 262, 3.1.2.6)

$$\varphi(t,t_0) = \varphi_{RH} \cdot \beta_{\sigma c} \cdot \beta_{fc} \cdot \beta(t_0) \cdot \beta(t-t_0) \qquad (\approx 1.5\ldots 2.5)$$

$\beta(t_0)$: Concrete age at loading



φ_{RH} : Load duration (→ progression)

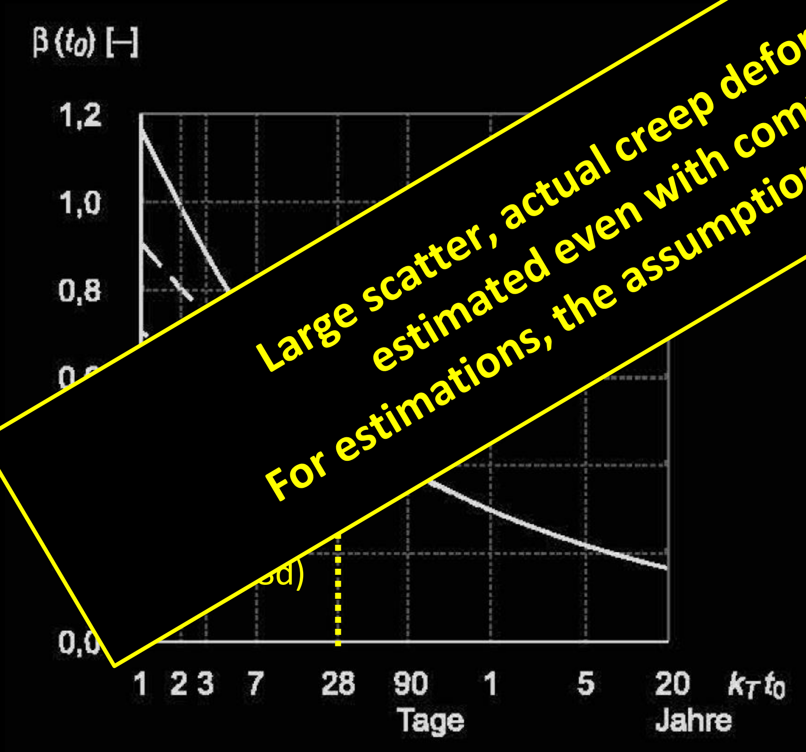


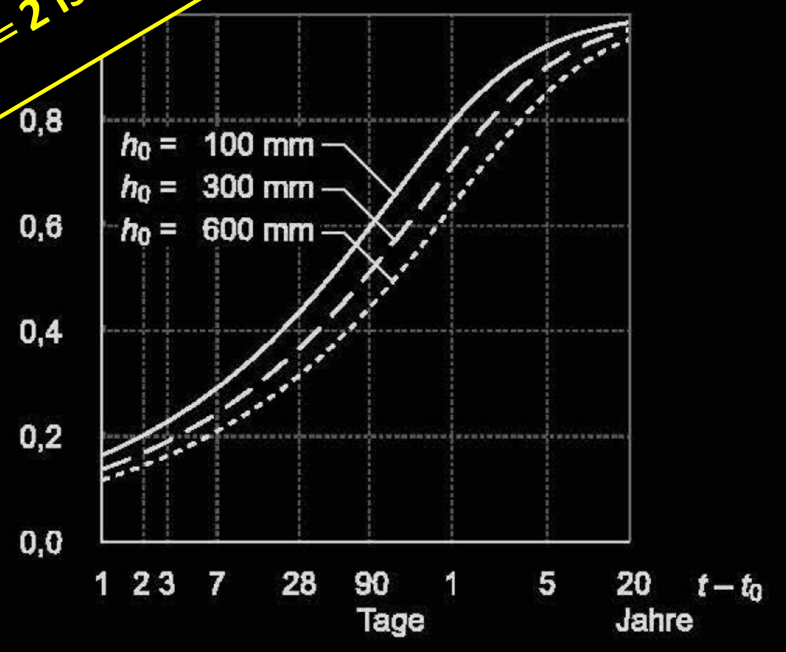
Long-term effects

Creep – Final value and progression
(see also SIA 262, 3.1.2.6)

$$\varphi\left(t,t_0\right)=\varphi_{RH}\cdot\beta_{\sigma c}\cdot\beta_{fc}\cdot\beta(t_0)\cdot \ldots \; 1.5\ldots 2.5)$$

$\beta(t_0)$: Concrete age at loading … (→ progression)





Large scatter, actual creep deformations can only be estimated even with complex calculations!
For estimations, the assumption $\varphi = 2$ is usually sufficient.

4 Long-term effects

4.2 Effect of creep on the load-bearing and deformation behaviour

Long-term effects

Effect of creep on deformations of a structure

- The effect of creep must always be taken into account when determining deformations due to permanent loads. The increase in deformation due to creep is considerably smaller in the cracked stage II than in the non-cracked stage I (see Stahlbeton I).
- Deformations are often governing the design, for example in the case of:
  - passively reinforced, slender girders (above $h/L \approx 1/12$)
  - passively reinforced slabs (flat slabs, canopies, slabs near facade area, non-load-bearing walls)
  - prestressed bridge girders, whose stresses in construction and final state differ strongly (cantilever construction, continuous beams cast span by span)

Effect of creep on internal forces and stresses

- Restraint and residual stresses are partially relieved due to creep over time (relaxation).
- For statically determinate (= isostatic) systems and for statically indeterminate systems with uniform creep properties, creep has no effect on the internal forces
- Significant internal force redistributions occur in statically indeterminate (= hyperstatic) systems as a result of changes in the static system and non-uniform creep properties.
  The calculation of creep effects is complicated by the interdependence (creep depends on the level of stress and vice versa).

Approaches for the calculation of creep and shrinkage problems

- Method with age-adjusted modulus of elasticity
- Unit creep curve method (Dischinger method) → see Appendix
- Rüsch Method (improved Dischinger method) → see Appendix
- Creep step method → see Appendix
- Trost Method (sufficiently accurate and suitable for manual calculations)

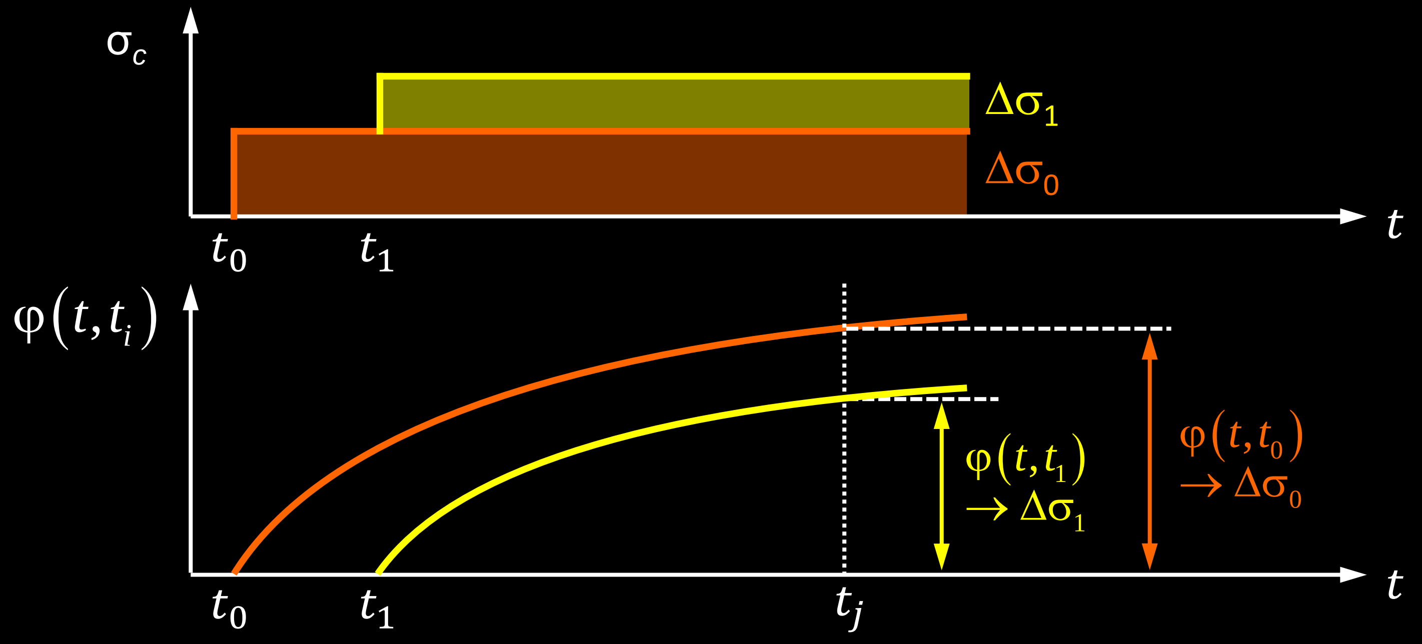
Long-term effects

Creep – Boltzmann superposition principle

- The creep strain due to any stress development $\sigma(t)$ can generally be expressed as follows:

$$\varepsilon_{cc}(t) = \frac{1}{E_{c0}} \int_{\tau=0}^{\tau=t} \frac{\partial \sigma}{\partial \tau} \varphi(t,\tau)\, d\tau$$

- For discrete stress increments (steps) $\Delta\sigma_i$, which are applied at time t_i results: $\varepsilon_{cc}(t) = \frac{1}{E_{c0}} \sum_{i=0}^{n} \Delta\sigma_i \cdot \varphi(t,t_i)$

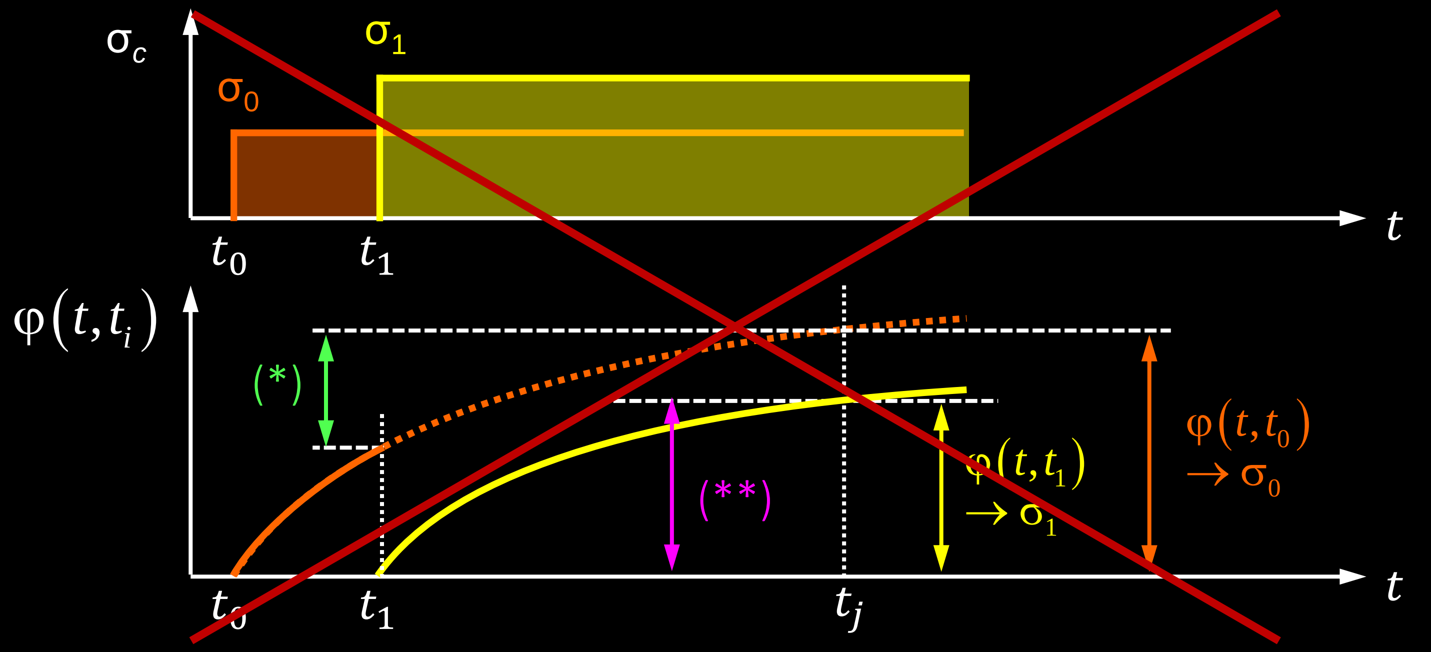


Long-term effects

Creep – Boltzmann superposition principle

Incorrect procedure for determining creep deformations (creep from the respective load level for the entire load with new creep coefficient):

- (*) Effective = correct portion of creep caused by σ_0 in time interval $t_1 \ldots t_j$ $\quad \Delta\varphi(t_j, t_1) \rightarrow \sigma_0$ (right)
- (**) Incorrectly determined portion of creep caused by σ_0 in time interval $t_1 \ldots t_j$ $\quad \Delta\varphi(t_j, t_1) \rightarrow \sigma_0$ (wrong)



4 Long-term effects

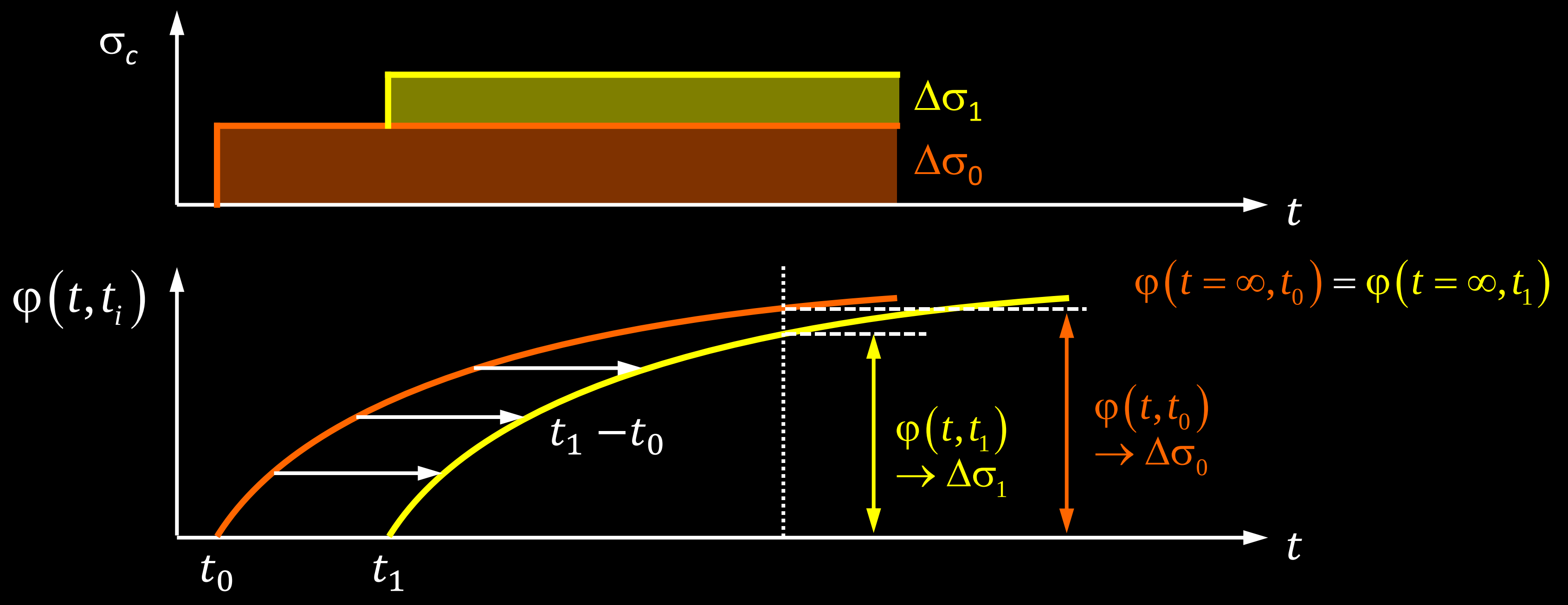
4.3 Simplified method for the investigation of long-term effects

Long-term effects

Approaches for the calculation of creep and shrinkage problems

Method with age-adjusted modulus of elasticity

- Effect of concrete age at loading neglected
  → same creep curve for all loads, shifted along the abscissa (horizontal)
- Unrealistic (overestimates the ability of old concrete to creep)

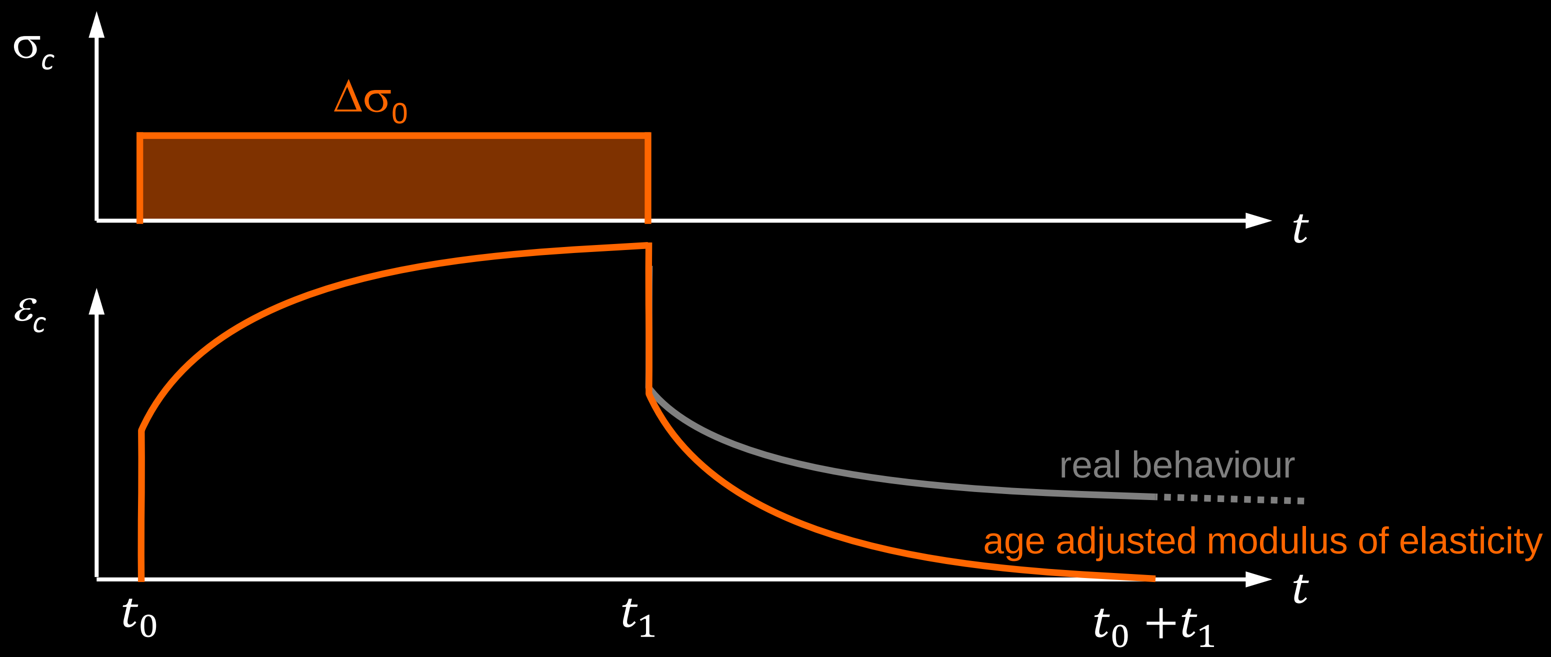


Long-term effects

Approaches for the calculation of creep and shrinkage problems

Method with age-adjusted modulus of elasticity

- Effect of concrete age at loading neglected
  → same creep curve for all loads, shifted along the abscissa (horizontal)
- Unrealistic (overestimates the ability of old concrete to creep)
- Unrealistic: corresponds to the assumption of viscoelastic, i.e. fully reversible behaviour



Long-term effects

Approaches for the calculation of creep and shrinkage problems

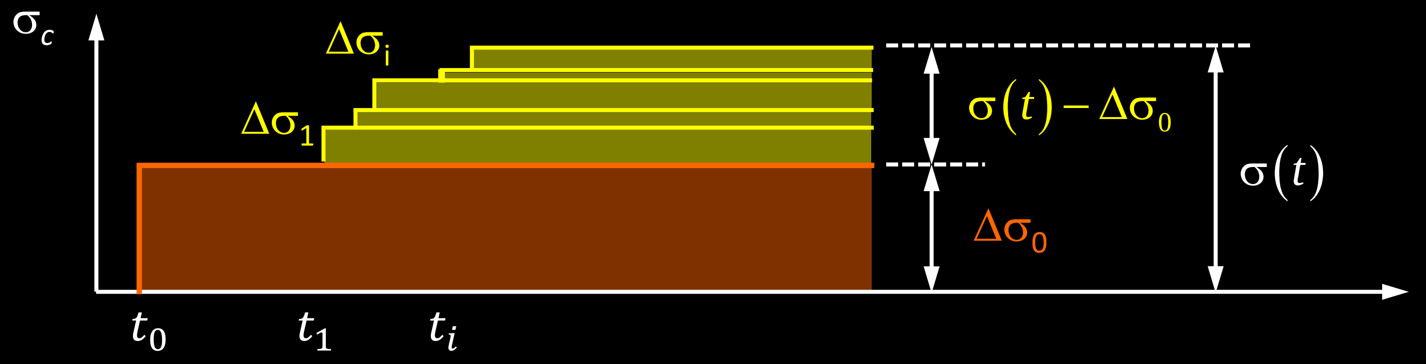
Trost method

In practice, a relatively large proportion of the total stress is applied at time t_0 followed by smaller stress increments $\Delta\sigma_i$ (additional loads, but also internal force redistributions). The Trost method takes advantage of this to avoid an iterative or step-by-step approach.

The creep function for the stress increments ($\sigma(t) - \Delta\sigma_0 = \sum_{i=1}^{n} \Delta\sigma_i$) occurring at the time period $t_i > t_0$ (resp. $t_0 < t_i \leq \infty$) is reduced with an ageing coefficient $\mu(t)$ (sometimes also called «relaxation factor»).

The creep deformation due to the total change in stress according to Boltzmann's superposition principle is:

$$\varepsilon_{cc}(t) = \frac{\sigma_0}{E_{c0}} \cdot \varphi(t, t_0) + \sum_{i=1}^{n} \frac{\Delta\sigma_i}{E_{c0}} \cdot \varphi(t, t_i) = \frac{\sigma_0}{E_{c0}} \cdot \varphi(t, t_0) + \frac{\sigma(t) - \sigma_0}{E_{c0}} \cdot \mu(t) \cdot \varphi(t, t_0)$$



Ageing coefficient (simplified by Trost to be identical for all load levels, $t_i > t_0$)
General derivation see Marti, Theory of Structures, Chapter 7.4.2

Long-term effects

Approaches for the calculation of creep and shrinkage problems

Trost method

The ageing coefficient results from the equation on the previous slide:

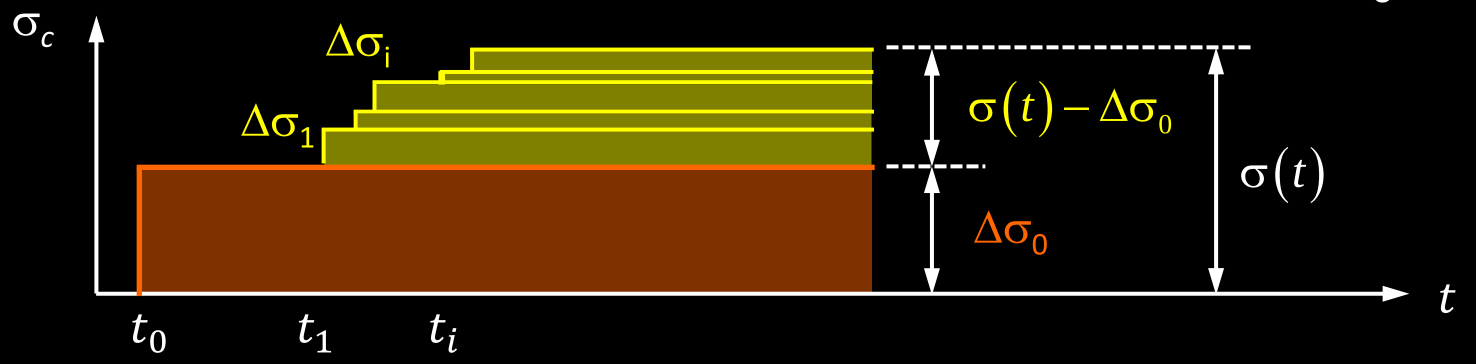
$$\sum_{i=1}^{n} \frac{\Delta\sigma_i}{E_{c0}} \cdot \varphi(t,t_i) = \frac{\sigma(t)-\sigma_0}{E_{c0}} \cdot \mu(t) \cdot \varphi(t,t_0) \rightarrow \mu(t) = \frac{\sum_{i=1}^{n} \Delta\sigma_i \cdot \varphi(t,t_i)}{(\sigma(t)-\sigma_0)\cdot\varphi(t,t_0)}$$

The total deformations at time *t* thus amount to:

$$\varepsilon_c(t) = \frac{\sigma_0}{E_{c0}}\left(1+\varphi(t,t_0)\right) + \frac{\sigma(t)-\sigma_0}{E_{c0}}\left(1+\mu(t)\cdot\varphi(t,t_0)\right) + \varepsilon_{cs}(t)$$

elastic deformations

shrinkage

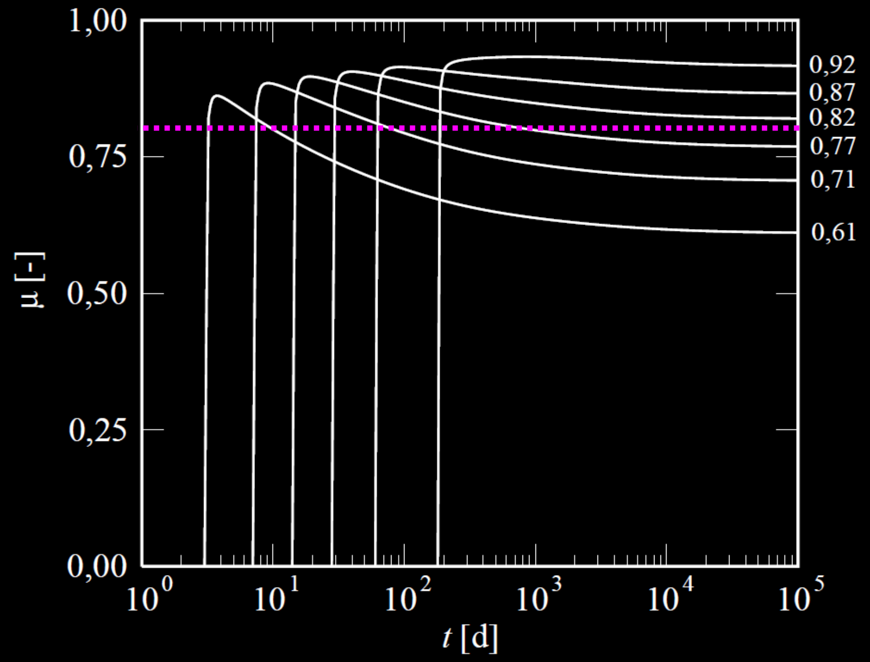


Long-term effects

Approaches for the calculation of creep and shrinkage problems

Trost method

- The stress curve is generally not known → $\mu(t)$ cannot be calculated directly in the way outlined on the previous slides
- If the relaxation function is determined from the creep function (solution of a linear, inhomogeneous Volterra integral equation), the corresponding ageing coefficient can be determined numerically [see Seelhofer 2009 or Marti, Theory of Structures]:



- The evaluation shows that $\mu(t)$ varies only slightly

→ Ageing coefficient μ is independent of time
sufficiently accurate for practical applications

→ for usual conditions (φ = 1.5...4)
approximately $\mu \approx 0.80$

Long-term effects

Approaches for the calculation of creep and shrinkage problems

Trost method

- With this approximation the total deformation at time *t* is:

$$\varepsilon_c(t) = \frac{1}{E_{c0}}\Big[\underbrace{\sigma_0(1+\varphi)}_{\text{initial stress}} + \underbrace{\Delta\sigma(1+\mu\cdot\varphi)}_{\text{stresses added over time}}\Big] + \varepsilon_{cs}(t)$$

with $\sigma_0 = \Delta\sigma_0 = \sigma(t=t_0), \quad \Delta\sigma = \sigma(t) - \sigma_0, \quad \varphi = \varphi(t,t_0), \quad t > t_{0,} \quad \mu \approx 0.8$

- Alternative formulation using fictitious («refined age-adjusted») moduli of elasticity for long-term influences:

$$\varepsilon_c(t) = \frac{\sigma_0}{\dfrac{E_{c0}}{1+\varphi(t,t_0)}} + \frac{\Delta\sigma(t)}{\dfrac{E_{c0}}{1+\mu\cdot\varphi(t,t_0)}} + \varepsilon_{cs}(t) = \frac{\sigma_0}{E'_c} + \frac{\Delta\sigma(t)}{E''_c} + \varepsilon_{cs}(t) : \underbrace{E'_c = \frac{E_{c0}}{1+\varphi(t,t_0)}}_{\text{initial stress}}, \quad \underbrace{E'' = \frac{E_{c0}}{1+\mu\cdot\varphi(t,t_0)}}_{\text{stresses added over time}}$$

Long-term effects

Calculation of relaxation function based on creep coefficient and ageing factor

- Relaxation function = stress curve for constant (imposed) initial strain, i.e. initial strains $\varepsilon_{c0} = \dfrac{\sigma_0}{E_{c0}}$ remain constant

Method with age adjusted modulus of elasticity $(\varphi = \varphi(t, t_0))$

$$\varepsilon_c(t) = \frac{\sigma_0}{E_{c0}}(1+\varphi) + \frac{\Delta\sigma(t)}{E_{c0}}(1+\varphi) = \frac{\sigma_0}{E_{c0}}$$

$$\rightarrow \Delta\sigma(t) = -\sigma_0 \frac{\varphi}{1+\varphi}$$

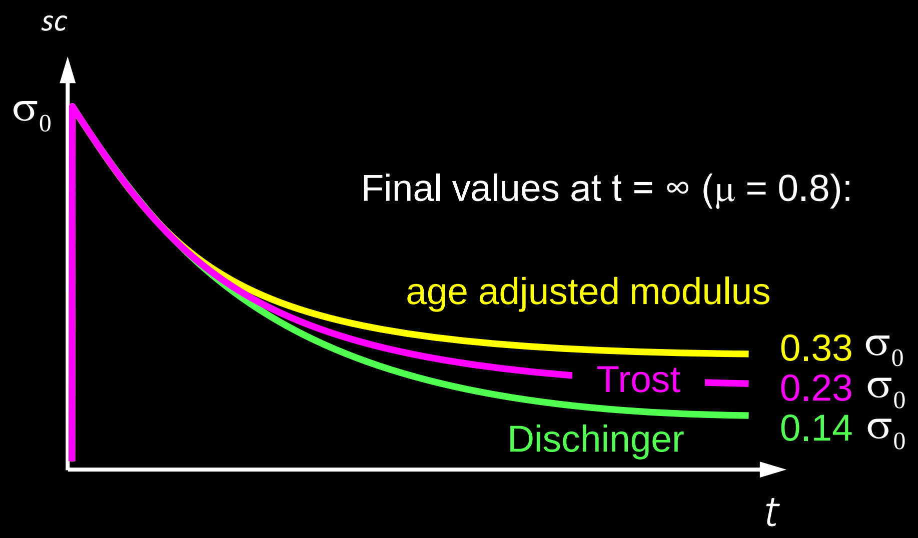
$$\rightarrow \sigma(t) = \sigma_0 + \Delta\sigma(t) = \sigma_0\left(1 - \frac{\varphi}{1+\varphi}\right) = \sigma_0 \frac{1}{1+\varphi}$$

Trost method $(\varphi = \varphi(t, t_0))$

$$\varepsilon_c(t) = \frac{\sigma_0}{E_{c0}}(1+\varphi) + \frac{\Delta\sigma(t)}{E_{c0}}(1+\mu\cdot\varphi) = \frac{\sigma_0}{E_{c0}}$$

$$\rightarrow \Delta\sigma(t) = -\sigma_0 \frac{\varphi}{1+\mu\cdot\varphi}$$

$$\rightarrow \sigma(t) = \sigma_0 + \Delta\sigma(t) = \sigma_0\left(1 - \frac{\varphi}{1+\mu\cdot\varphi}\right)$$



The Trost method is simple and agrees well with experiments (better than more complicated procedures) → only this procedure is used in the following!

4 Long-term effects

4.4 Examples

Long-term effects

Terminology and generalisation of the Force Method → Time-dependent Force Method

- In the following, Trost's Method is used in combination with the Force Method, known from «Baustatik I/II»
- To account for long-term effects, compatibility conditions are expressed here at different moments in time
- Further information on the Force Method: Peter Marti, «Theory of Structures» resp. «Baustatik», Chapter 16. The following summaries are taken from this book (p. 254 and p. 257, respectively):

1. Determine the degree n of static indeterminacy.
2. Select a stable, statically determinate *basic system* by releasing n constraints and introducing corresponding *redundant variables* X_i.
3. Determine the support force variables and stress resultants C_0, S_0 and C_i, S_i for the basic system as a result of loads or as a result of unit force variables $X_i = 1$.
4. Determine the deformations (incompatibilities) δ_{i0} or δ_{ij} at the position and in the direction of X_i as a result of the external actions (loads and imposed deformations) or as a result of the unit force variables $X_j = 1$.
5. Set up and solve the following *compatibility conditions*:

$$\delta_i = \delta_{i0} + \sum_{j=1}^{n} \delta_{ij} X_j = 0 \qquad (i = 1, 2, \ldots, n) \tag{16.8}$$

6. Determine the support force variables and stress resultants for the statically indeterminate system by *superposing* the corresponding variables on the basic system:

$$C = C_0 + \sum_{i=1}^{n} C_i X_i \quad , \quad S = S_0 + \sum_{i=1}^{n} S_i X_i \tag{16.9}$$

1. Bestimmen des Grads n der statischen Unbestimmtheit.
2. Wahl eines stabilen, statisch bestimmten *Grundsystems* durch Lösen von n Bindungen und Einführen entsprechender *überzähliger Grössen* X_i.
3. Ermitteln der Lagerkraft- und Schnittgrössen C_0, S_0 bzw. C_i, S_i am Grundsystem infolge der Lasten bzw. infolge der Einheitskraftgrössen $X_i = 1$.
4. Ermitteln der Verformungen (Klaffungen) δ_{i0} bzw. δ_{ij} an der Stelle und in der Richtung von X_i infolge der äusseren Einwirkungen (Lasten und eingeprägte Verformungen) bzw. infolge der Einheitskraftgrössen $X_j = 1$.
5. Aufstellen und Lösen der *Kompatibilitätsbedingungen*

$$\delta_i = \delta_{i0} + \sum_{j=1}^{n} \delta_{ij} X_j = 0 \qquad (i = 1, 2, \ldots, n) \tag{16.8}$$

6. Bestimmen der Lagerkraft- und Schnittgrössen des statisch unbestimmten Systems durch *Superposition* der entsprechenden Grössen am Grundsystem

$$C = C_0 + \sum_{i=1}^{n} C_i X_i \quad , \quad S = S_0 + \sum_{i=1}^{n} S_i X_i \tag{16.9}$$

Long-term effects

Redistribution of internal forces in statically indeterminate systems

Systems with uniform creep properties

Example 1: Two-span beam, solution with force method

BS (basic system): Intermediate bearing removed
RV: (redundant variable): Reaction at intermediate support

Displacements in the basic system (elastic, $t = t_0$):

$$\delta_{10} = \frac{5}{384}\frac{g_k(2l)^4}{EI} \qquad \delta_{11} = \frac{1}{48}\frac{(2l)^3}{EI}$$

Compatibility condition at time t_0:

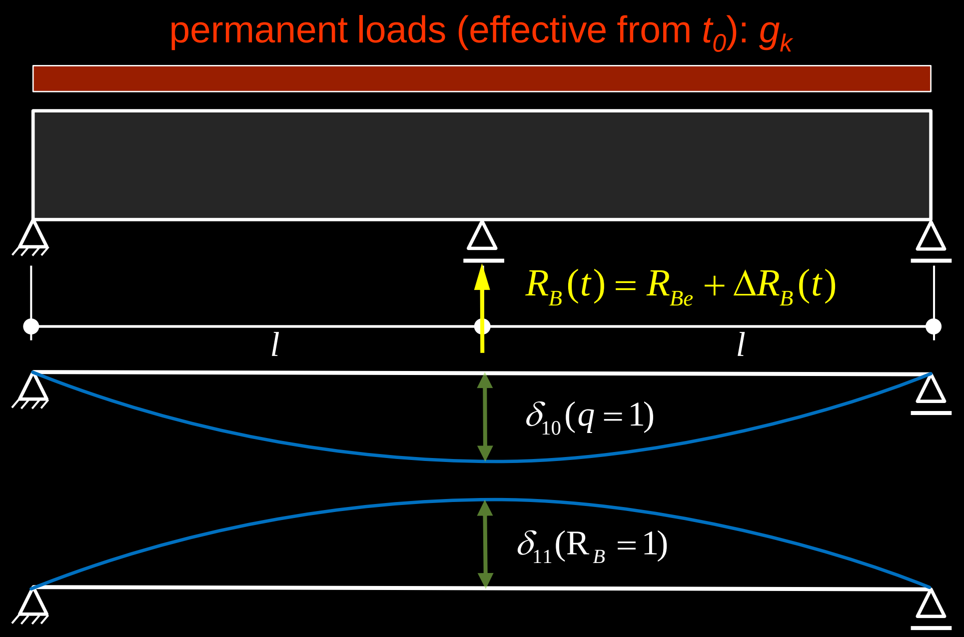
$$\delta = \delta_{10} + R_{Be} \cdot \delta_{11} = 0$$

Time-dependent compatibility condition with the Trost method:

$$\delta = \delta_{10} \cdot (1+\varphi) + R_{Be} \cdot \delta_{11} \cdot (1+\varphi) + \Delta R_B(t) \cdot \delta_{11} \cdot (1+\mu\varphi) = 0$$

$$\underbrace{\delta_{10} + R_{Be} \cdot \delta_{11}}_{= 0 \text{ (compatibility at time } t_0)} + \Delta R_B(t) \cdot \delta_{11} \frac{1+\mu\varphi}{1+\varphi} = 0 \quad \boxed{\rightarrow \Delta R_B(t) = 0}$$

= 0 (compatibility at time t_0)



→ Generalization to general systems is possible → With uniform creep properties, the redundant variables of stat. indeterminate systems do not change!

Long-term effects

Redistribution of internal forces in statically indeterminate systems

Systems with uniform creep properties

Generalization to general systems

Compatibility at $t = t_0$:
$$\begin{pmatrix} \delta_{10} \\ \vdots \\ \delta_{i0} \end{pmatrix} + \begin{pmatrix} \delta_{11} & \cdots & \delta_{1i} \\ \vdots & \ddots & \vdots \\ \delta_{i1} & \cdots & \delta_{ii} \end{pmatrix} \cdot \begin{pmatrix} X_{1e} \\ \vdots \\ X_{ie} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad (**)$$

$\rightarrow$ RV at $t = t_0$:
$$\begin{pmatrix} X_{1e} \\ \vdots \\ X_{ie} \end{pmatrix} = \begin{pmatrix} \delta_{11} & \cdots & \delta_{1i} \\ \vdots & \ddots & \vdots \\ \delta_{i1} & \cdots & \delta_{ii} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \delta_{10} \\ \vdots \\ \delta_{i0} \end{pmatrix}$$

All coefficients multiplied by the same factor!

Change of redundant variables using the Trost method: $X_i(t) = X_{ie} + \Delta X_i(t)$

$\rightarrow$ Compatibility for $t > t_0$:
$$(1+\varphi)\begin{pmatrix} \delta_{10} \\ \vdots \\ \delta_{i0} \end{pmatrix} + \begin{pmatrix} \delta_{11} & \cdots & \delta_{1i} \\ \vdots & \ddots & \vdots \\ \delta_{i1} & \cdots & \delta_{ii} \end{pmatrix} \cdot \left[(1+\varphi)\begin{pmatrix} X_{1e} \\ \vdots \\ X_{ie} \end{pmatrix} + (1+\mu\varphi)\begin{pmatrix} \Delta X_1 \\ \vdots \\ \Delta X_i \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\rightarrow$ using (**):
$$\begin{pmatrix} \delta_{11} & \cdots & \delta_{1i} \\ \vdots & \ddots & \vdots \\ \delta_{i1} & \cdots & \delta_{ii} \end{pmatrix} \cdot \begin{pmatrix} \Delta X_1 \\ \vdots \\ \Delta X_i \end{pmatrix} \frac{(1+\mu\varphi)}{(1+\varphi)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{where} \quad \left\| \begin{matrix} \delta_{11} & \cdots & \delta_{1i} \\ \vdots & \ddots & \vdots \\ \delta_{i1} & \cdots & \delta_{ii} \end{matrix} \right\| \neq 0 \quad \rightarrow \begin{pmatrix} \Delta X_1 \\ \vdots \\ \Delta X_i \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

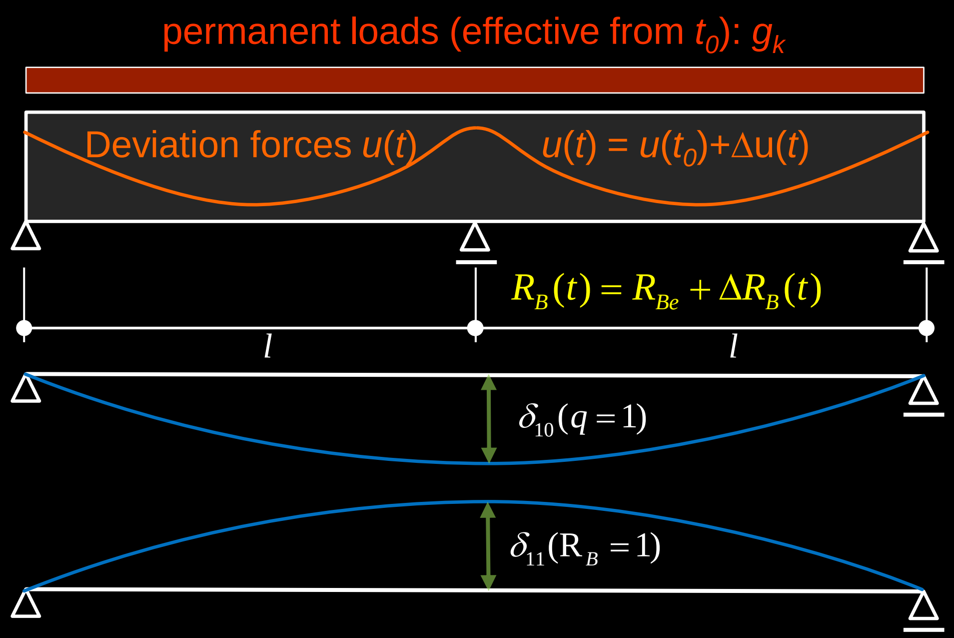
If the creep properties are uniform, redundant variables do not change in statically indeterminate systems!

Long-term effects

Redistribution of internal forces in statically indeterminate systems

Systems with uniform creep properties

Example 2: Prestressed two-span beam, solution with force method



BS: Intermediate bearing removed
RV: Reaction intermediate support

Displacements in the basic system (elastic, $t = t_0$):

$$\delta_{10} = \frac{5}{384}\frac{(2l)^4}{EI} \qquad \delta_{11} = \frac{1}{48}\frac{(2l)^3}{EI}$$

Compatibility condition at time t_0 :

$$\delta = (g_k + u(t_0))\cdot\delta_{10} + R_{Be}\cdot\delta_{11} = 0$$

Time-dependent compatibility condition with the Trost method:

$$\delta_B = (g_k + u(t_0))\cdot\delta_{10}(1+\varphi) + R_{Be}\cdot\delta_{11}(1+\varphi) + \Delta u(t)\cdot\delta_{10}(1+\mu\varphi) + \Delta R_B(t)\cdot\delta_{11}(1+\mu\varphi) = 0$$

$$(g_k + u(t_0))\cdot\delta_{10} + R_{Be}\cdot\delta_{11} + \Delta u(t)\cdot\delta_{10}\frac{1+\mu\varphi}{1+\varphi} + \Delta R_B(t)\cdot\delta_{11}\frac{1+\mu\varphi}{1+\varphi} = 0$$

= 0 (compatibility at time t_0)

$$\rightarrow \Delta R_B(t) = -\Delta u(t)\frac{\delta_{10}}{\delta_{11}}$$

→ Support reactions change due to time-dependent prestressing losses (RV proportional to prestressing force = deviation force)

Long-term effects

Redistribution of internal forces in statically indeterminate systems

Systems with non-uniform creep properties

Example 3: Hinged frame with concrete beam and steel columns, solution with force method

Displacements in the basic system (elastic, $t = t_0$):

$$\delta_{10}^{R} = -\frac{gl^2}{8}\cdot\frac{l}{4}\cdot\frac{2}{3}\cdot\frac{l}{EI_R} = -\frac{gl^4}{48EI} \qquad \delta_{10}^{S} = 0 \qquad \delta_{11}^{R} = -\left(\frac{l}{4}\right)^2\cdot\frac{l}{EI_R} = \frac{l^3}{16EI} \qquad \delta_{11}^{S} = 2\cdot\left(\frac{l}{4}\right)^2\cdot\frac{1}{3}\cdot\frac{h}{EI_S} = \frac{l^3}{16EI}$$

Compatibility condition at time t_0:

$$\delta_1(t_0) = \delta_{10}^{S} + \delta_{10}^{R} + X_{1e}\left(\delta_{11}^{S} + \delta_{11}^{R}\right) = 0 \rightarrow X_{1e} = -\frac{\delta_{10}^{S} + \delta_{10}^{R}}{\delta_{11}^{S} + \delta_{11}^{R}} = \frac{g_k l}{6}$$

Time-dependent compatibility condition with the Trost method (support does not creep), taking into account the compatibility at time t_0:

$$\delta_1(t) = \delta_{10}^{S} + \delta_{10}^{R}(1+\varphi) + X_{1e}\left[\delta_{11}^{S} + \delta_{11}^{R}(1+\varphi)\right] + \Delta X_1\left[\delta_{11}^{S} + \delta_{11}^{R}(1+\mu\varphi)\right] = 0$$

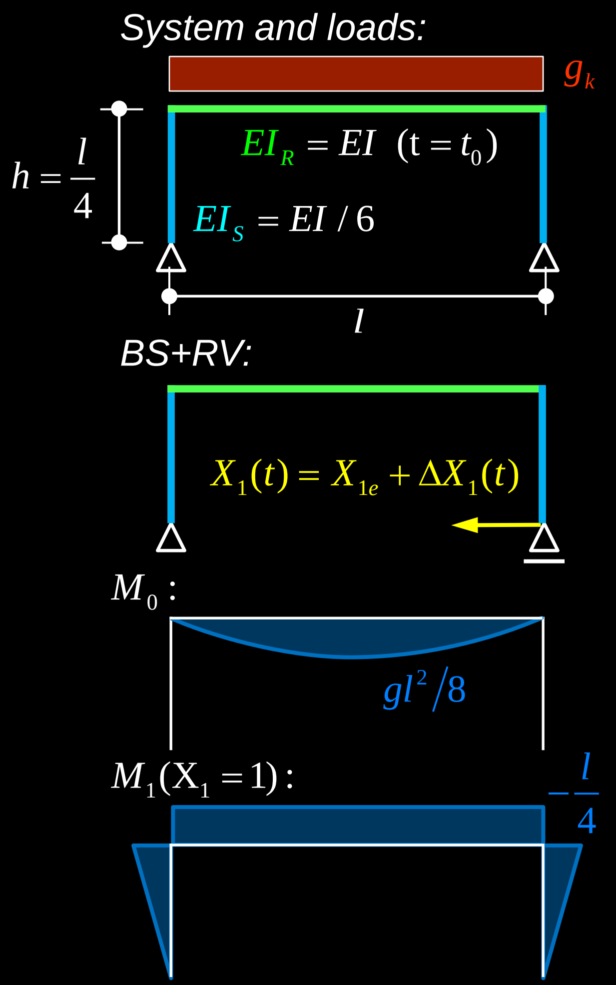
$$\delta_{10}^{S} + \delta_{10}^{R} + \delta_{10}^{R}\cdot\varphi + X_{1e}\left(\delta_{11}^{S} + \delta_{11}^{R}\right) + X_{1e}\cdot\delta_{11}^{R}\cdot\varphi + \Delta X_1\left[\delta_{11}^{S} + \delta_{11}^{R}(1+\mu\varphi)\right] = 0$$

$$\delta_{10}^{R}\varphi + X_{1e}\cdot\delta_{11}^{R}\varphi + \Delta X_1\left[\delta_{11}^{S} + \delta_{11}^{R}(1+\mu\varphi)\right] = 0 \rightarrow \Delta X_1(t) = -\varphi\frac{\delta_{10}^{R} + X_{1e}\cdot\delta_{11}^{R}}{\delta_{11}^{S} + \delta_{11}^{R}(1+\mu\varphi)}$$

$$\rightarrow \Delta X_1(t) = \frac{g_k l}{6}\,\frac{\varphi}{2+\mu\varphi} = X_{1e}\,\frac{\varphi}{2+\mu\varphi}, \qquad \boxed{X_1(t) = \frac{g_k l}{6}\left(1+\frac{\varphi}{2+\mu\varphi}\right)}$$

Formula applies only for this example (system-dependent)

→ In the case of non-uniform creep properties internal forces are redistributed due to creep



Long-term effects

Redistribution of internal forces in statically indeterminate systems

Systems with non-uniform creep properties - Generalization to general systems

Compatibility at $t = t_0$:
$$\begin{pmatrix}\delta_{10}\\ \vdots \\ \delta_{i0}\end{pmatrix} + \begin{pmatrix}\delta_{11} & \cdots & \delta_{1i}\\ \vdots & \ddots & \vdots \\ \delta_{i1} & \cdots & \delta_{ii}\end{pmatrix} \cdot \begin{pmatrix}X_{1e}\\ \vdots \\ X_{ie}\end{pmatrix} = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix} \quad (**)$$

$\rightarrow$ RV at $t = t_0$:
$$\begin{pmatrix}X_{1e}\\ \vdots \\ X_{ie}\end{pmatrix} = \begin{pmatrix}\delta_{11} & \cdots & \delta_{1i}\\ \vdots & \ddots & \vdots \\ \delta_{i1} & \cdots & \delta_{ii}\end{pmatrix}^{-1} \cdot \begin{pmatrix}\delta_{10}\\ \vdots \\ \delta_{i0}\end{pmatrix}$$

no constant factor for creep influence
(each RV has a different factor in general)
$\rightarrow$ redundant variables must change!

Change of redundant variables using the Trost method: $X_i(t) = X_{ie} + \Delta X_i(t)$

$\rightarrow$ Compatibility for $t > t_0$:
$$(1+\varphi)\begin{pmatrix}\delta_{10}\\ \vdots \\ \delta_{i0}\end{pmatrix} + \begin{pmatrix}\delta_{11} & \cdots & \delta_{1i}\\ \vdots & \ddots & \vdots \\ \delta_{i1} & \cdots & \delta_{ii}\end{pmatrix} \cdot \left[(1+\varphi)\begin{pmatrix}X_{1e}\\ \vdots \\ X_{ie}\end{pmatrix} + (1+\mu\varphi)\begin{pmatrix}\Delta X_1\\ \vdots \\ \Delta X_i\end{pmatrix}\right] = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}$$

$\rightarrow$ using (**):
$$\begin{pmatrix}\delta_{11} & \cdots & \delta_{1i}\\ \vdots & \ddots & \vdots \\ \delta_{i1} & \cdots & \delta_{ii}\end{pmatrix} \cdot \begin{pmatrix}\Delta X_1\\ \vdots \\ \Delta X_i\end{pmatrix} \frac{(1+\mu\varphi)}{(1+\varphi)} = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix} \quad \text{where} \quad \left|\begin{pmatrix}\delta_{11} & \cdots & \delta_{1i}\\ \vdots & \ddots & \vdots \\ \delta_{i1} & \cdots & \delta_{ii}\end{pmatrix}\right| \neq 0 \quad \rightarrow \begin{pmatrix}\Delta X_1\\ \vdots \\ \Delta X_i\end{pmatrix} = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}$$

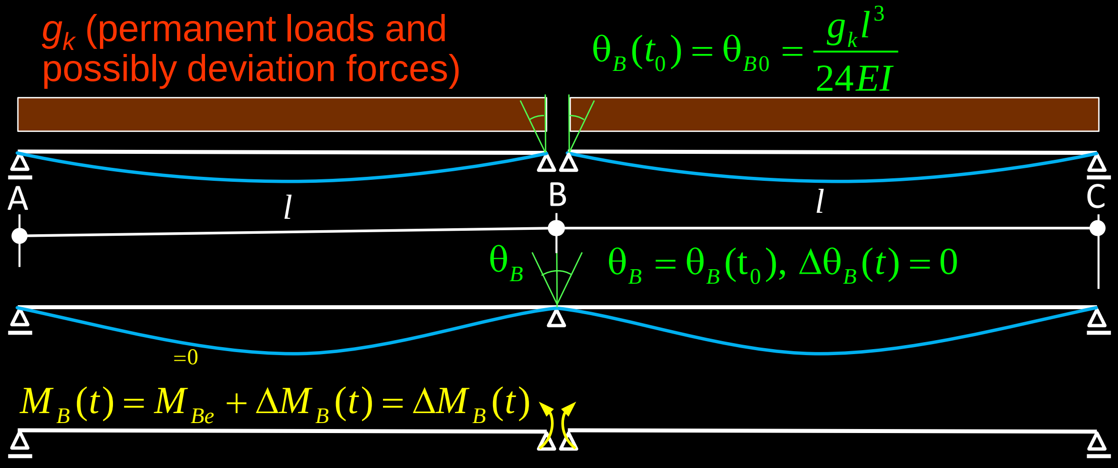
$\rightarrow$ In the case of non-uniform creep properties, internal force redistributions occur as a result of creep

Long-term effects

Influence of creep for system changes

Example 4 - Connection of two simple beams with the same creep behaviour

System, load relevant for creep:



Construction sequence:

1. Two single span girders are positioned (lifted in)
2. $t = t_0$: Monolithic connection at B

BS+RV: $\theta_{B0} = \frac{g_k l^3}{24EI}, \quad \theta_{B1} = \frac{l}{3EI}$

Bending moment and girder end rotation at B (per side) at t_0:

$$M_B(t_0) = M_{Be} = 0, \quad \theta_B(t_0) = \theta_{B0} = \frac{g_k l^3}{24EI}$$

Comparison: Bending moment at B on a single two-span beam: («OC»: one casting)

$$M_{B,OC} = -\frac{\theta_{B0}}{\theta_{B1}} = -\frac{g_k l^2}{8}$$

Compatibility condition (relative rotation of girder ends at B remains constant after $= t_0$):

$$\Delta\theta_B(t) = \theta_B(t) = \theta_{B0} \cdot \varphi + \Delta M_B(t) \cdot \theta_{B1}\left(1 + \mu\varphi\right) = 0$$

$$\rightarrow \Delta M_B(t) = -\frac{\theta_{B0}}{\theta_{B1}} \frac{\varphi}{1+\mu\varphi} = \boxed{M_{B,OC} \cdot \frac{\varphi}{1+\mu\varphi} = M_B(t)}$$

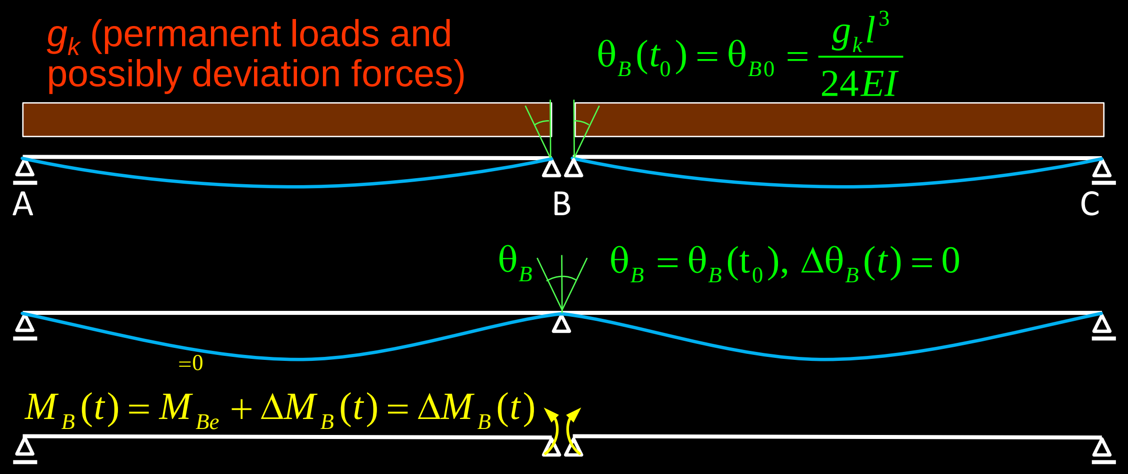
At the intermediate support, a moment of approx. 80% of the two-span beam built in one casting «OC» develops due to creep.

Long-term effects

Influence of creep in system changes

The ratio of the moment at B to the moment of the system built in one casting «OC» for various points in time and creep coefficients:

	56 days	180 days	1 year	5 years
φ(t)	1.00	1.75	2.00	2.50
$M_B(t)/M_{B,OC}$	0.56	0.73	0.77	0.83



As a general rule, in system changes, creep largely builds up the stress state of the system built in one casting σ_{OC}. The higher the creep coefficient, the closer it approximates the state of the system built in one casting («Einguss-System» in German).

As an approximation one may use:

φ ≈ 1 φ ≈ 2

$$S_{t=\infty} \approx S_A + (0.6\ldots0.8)(S_{OC} - S_A)$$

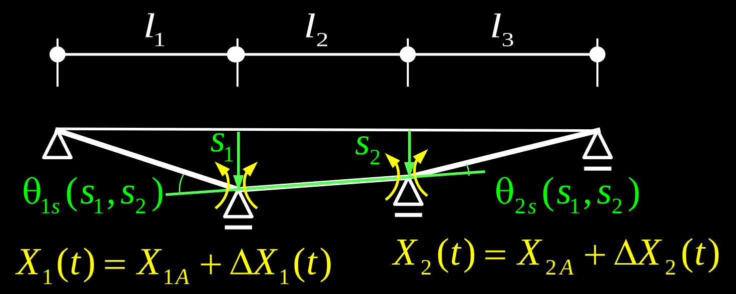
S_A Internal forces before system change (initial state)

S_{OC} Internal forces of system built in one casting "OC"

Long-term effects

Influence of creep on restraint internal forces (caused by imposed deformations)

Example 5a - three-span beam, time-independent («fast») support displacements s_1, s_2



Compatibility condition at time $t = t_0$:

$$\begin{matrix} X_{1A}\theta_{11} + X_{2A}\theta_{12} = \theta_{1s} \\ X_{1A}\theta_{21} + X_{2A}\theta_{22} = \theta_{2s} \end{matrix} \rightarrow \begin{pmatrix} X_{1A} \\ X_{2A} \end{pmatrix} = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}^{-1} \begin{pmatrix} \theta_{1s} \\ \theta_{2s} \end{pmatrix}$$

Time-dependent compatibility condition (Trost method):

$$\Delta\theta_1(t) = X_{1A}\theta_{11} \cdot \varphi + \Delta X_1(t)\theta_{11} \cdot (1+\mu\varphi) + X_{2A}\theta_{12} \cdot \varphi + \Delta X_2(t)\theta_{12} \cdot (1+\mu\varphi) = 0$$
$$\Delta\theta_2(t) = X_{1A}\theta_{21} \cdot \varphi + \Delta X_1(t)\theta_{21} \cdot (1+\mu\varphi) + X_{2A}\theta_{22} \cdot \varphi + \Delta X_2(t)\theta_{22} \cdot (1+\mu\varphi) = 0$$

	56 days	5 years
φ(t)	1.00	2.00
$X_i(t)/X_{iA}(t)$	0.44	0.23

...ditto, inversely:

$$\begin{matrix} \Delta X_1(t)\theta_{11} + \Delta X_2(t)\theta_{12} = -\overbrace{\left[X_{1A}\theta_{11} + X_{2A}\theta_{12}\right]}^{\theta_{1s}}\dfrac{\varphi}{1+\mu\varphi} \\ \Delta X_1(t)\theta_{21} + \Delta X_2(t)\theta_{22} = -\underbrace{\left[X_{1A}\theta_{21} + X_{2A}\theta_{22}\right]}_{\theta_{2s}}\dfrac{\varphi}{1+\mu\varphi} \end{matrix} \rightarrow \begin{pmatrix} \Delta X_1(t) \\ \Delta X_2(t) \end{pmatrix} = -\frac{\varphi}{1+\mu\varphi} \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}^{-1} \begin{pmatrix} \theta_{1s} \\ \theta_{2s} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \Delta X_1(t) \\ \Delta X_2(t) \end{pmatrix} = -\frac{\varphi}{1+\mu\varphi} \begin{pmatrix} X_{1A} \\ X_{2A} \end{pmatrix} \quad \text{resp.} \quad \boxed{X_i(t) = X_{iA}\left(1 - \frac{\varphi}{1+\mu\varphi}\right)}$$

(analogous to relaxation function)

→ Time-independent restraint forces ("fast imposed deformation") are reduced by creep (or relaxation) to 1/3...1/4 of the initial value

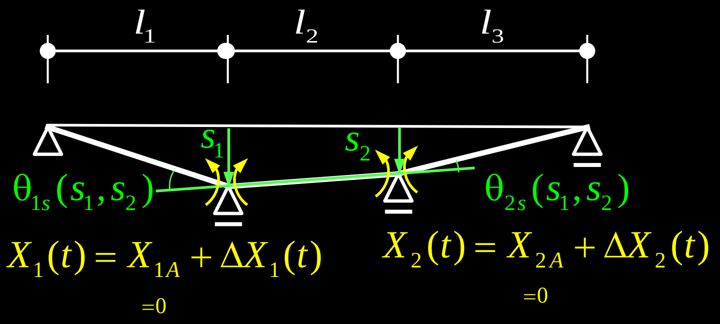
Long-term effects

Influence of creep on restraint internal forces (caused by imposed deformations)

Example 5b - three-span beam, time-dependent («slow») support displacements s_1, s_2

Assumption: Settlement process (s_1, s_2) proportional to creep function:

$$s_i(t) = s_i(t=\infty)\frac{\varphi(t,t_0)}{\varphi(t=\infty,t_0)} = s_{i,\infty}\frac{\varphi}{\varphi_\infty} \qquad t=t_0: \quad \begin{matrix} s_i = 0 \\ X_i = 0 \end{matrix}$$



Time-dependent compatibility condition (Trost method):

$$\Delta\theta_1(t) = \Delta X_1(t)\theta_{11}\cdot(1+\mu\varphi) + \Delta X_2(t)\theta_{12}\cdot(1+\mu\varphi) = \theta_{1s,\infty}\frac{\varphi}{\varphi_\infty}$$

$$\Delta\theta_2(t) = \Delta X_1(t)\theta_{21}\cdot(1+\mu\varphi) + \Delta X_2(t)\theta_{22}\cdot(1+\mu\varphi) = \theta_{2s,\infty}\frac{\varphi}{\varphi_\infty}$$

	56 days	180 days	5 years
φ(t)	1.00	1.75	2.00
$X_i(t)/X_{iE,el}(t)$	0.44	0.36	0.38

...ditto, inversely:

$$\begin{matrix} \Delta X_1(t)\theta_{11} + \Delta X_2(t)\theta_{12} = \theta_{1s,\infty}\dfrac{\varphi}{\varphi_\infty(1+\mu\varphi)} \\ \Delta X_1(t)\theta_{21} + \Delta X_2(t)\theta_{22} = \theta_{2s,\infty}\dfrac{\varphi}{\varphi_\infty(1+\mu\varphi)} \end{matrix} \rightarrow \begin{pmatrix} \Delta X_1(t) \\ \Delta X_2(t) \end{pmatrix} = \frac{\varphi}{\varphi_\infty(1+\mu\varphi)}\begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}^{-1}\begin{pmatrix} \theta_{1s,\infty} \\ \theta_{2s,\infty} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \frac{\varphi}{\varphi_\infty(1+\mu\varphi)}\begin{pmatrix} X_{1E,el} \\ X_{2E,el} \end{pmatrix} \text{ resp. } \boxed{X_i(t) = X_{iE,el}\frac{\varphi}{\varphi_\infty(1+\mu\varphi)}}$$

$X_{iE,el}$: value in elastic system subjected to $\theta_{is,\infty}$ without creep

→ Due to creep (or relaxation) time-dependent restraint forces ("slow imposed deformation") reach only approx. 40% of the elastic (short-term) value

4 Long-term effects

4.5 Final Remarks and Summary

Long-term effects

Aspects not covered in the lecture:

Composite cross-sections of concrete and steel or precast concrete components and in-situ concrete

→ Residual stresses or force redistributions due to shrinkage and creep of the concrete
(steel does neither creep nor shrink, prefabricated components creep less than in-situ concrete)

→ Determination of the force redistributions from the compatibility condition (plane cross-sections remain plane)

→ Consideration of creep due to time-dependent residual stresses with the Trost method

Effect of crack formation on creep behaviour

→ In all previous slides, uncracked behaviour was assumed (results valid e.g. for girders fully prestressed under permanent loads)

→ Crack formation and long-term effects influence one another

→ Approximate calculation analogous to the non-cracked state with fictitious creep coefficient φ':
- Determination of cracked elastic stiffness $EI^{II}_{t=0}$ with E_{c0} resp. $EI^{II}_{t=\infty}$ with $E_{c0}/(1+\varphi)$ (see Stahlbeton I)
- Calculation with $EI^{II}_{t=0}$ using the fictitious creep coefficient $\varphi' = EI^{II}_{t=0} / EI^{II}_{t=\infty} - 1$.

Effect of creep on prestressed systems

→ Prestress losses due to shrinkage, creep, and relaxation of the prestressing steel see Stahlbeton II.

→ Internal forces due to prestressing are to be taken into account when determining the creep-generating stresses. Treatment as anchorage, deviation, and friction forces (prestressing on the load side) is advisable → Creep is caused by sum of permanent loads and anchorage and deviation forces due to prestressing.

→ For highly prestressed, deformation-sensitive systems, such as cantilever bridges during the construction stage (*), the long-term effects must be carefully investigated, and upper/lower limit values must be used.

(*) large deformations due to dead weight (+) and prestressing (-), resulting deformation = difference, sensitive to assumptions made (there is no "safe side" when determining camber = «Überhöhung» in German)

Long-term effects

Summary

- The term "long-term effects" covers shrinkage, creep, and relaxation. Creep and relaxation of concrete are related phenomena.
- Due to the large variability of the material properties, the shrinkage and creep behaviour can only be determined approximately, even with complex calculations.
- All permanent actions (dead weight, superimposed loads, prestressing) cause creep.
- The stress history usually depends on the creep behaviour. The solution of creep problems, therefore, requires an iterative / step-by-step approach. For manual calculations, the Trost method (with an ageing coefficient of $\mu \approx 0.8$ for stresses that do not act from the beginning) is appropriate.
- In statically indeterminate systems with uniform creep properties, the restraint forces due to creep change exclusively as a result of time-dependent prestress losses (RV due to prestressing is proportional to the prestressing force resp. the deviation forces).
- In statically indeterminate systems with non-uniform creep properties, the redundant variables change as a result of creep.
- After system changes, creep largely builds up the stress state of the system built in one casting. The more prone to creep the system components are, the closer it approximates the system built in one casting. For normal conditions ($\varphi \approx 2$) approx. 80% of the latter is reached.
- Time-independent restraint forces ("fast imposed deformation") are reduced by creep (resp. relaxation) to 1/3...1/4 of the initial value. The reduction of the initial restraint forces is larger, the more prone to creep the system components are.

$$X_i(t) = X_{iA}\left(1 - \frac{\varphi}{1+\mu\varphi}\right)$$

- Time-dependent restraint forces ("slow imposed deformation") achieve as a result of creep (resp. relaxation) only approx. 40% of the elastic (short-term) value. The restraint forces never act in full-size and the more prone to creep the system parts are, the less they build up.

$$X_i(t) = X_{iE,el}\frac{\varphi}{\varphi_\infty(1+\mu\varphi)}$$

- Relaxation reduces the restraint forces, but not the deformations!

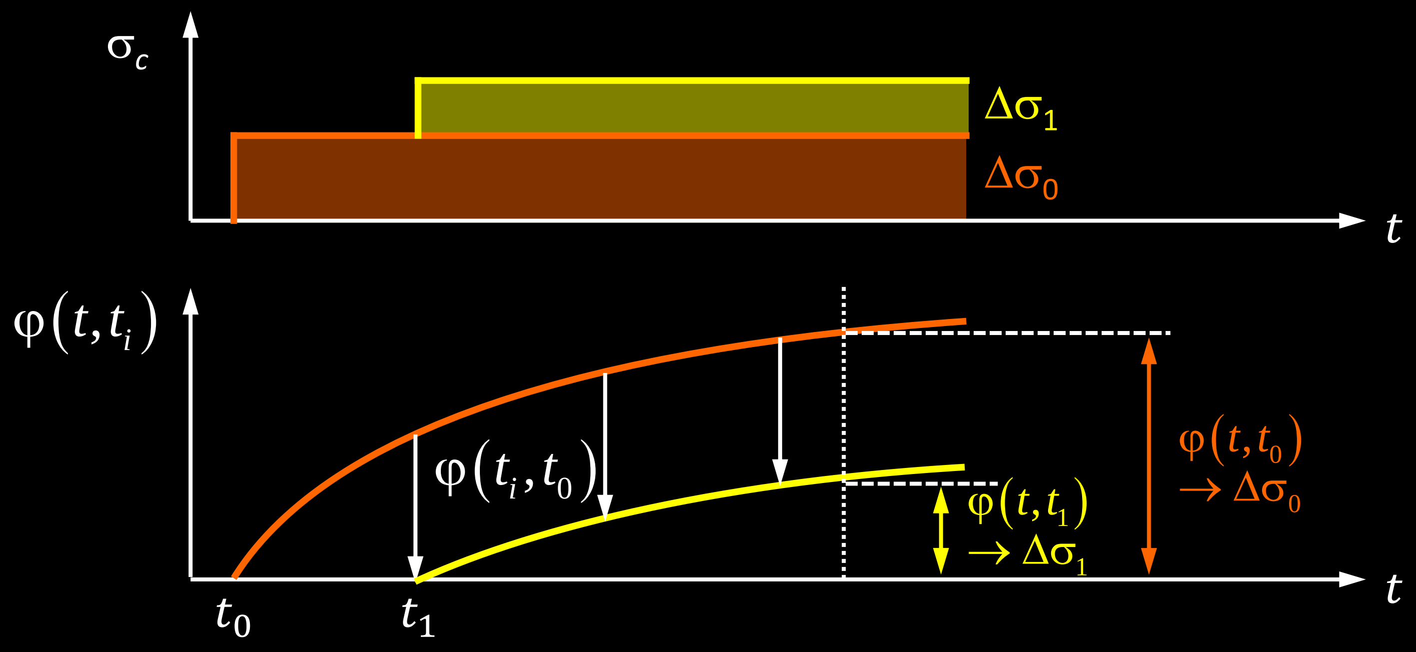
Appendix

Long-term effects

Approaches for the calculation of creep and shrinkage problems

Unit creep curve method (Dischinger method)

- Same creep curve for all loads, shifted along the ordinate (i.e. vertically)
- Advantage: Representation in recursion formulae possible
- Unrealistic: underestimates the creep of old concrete

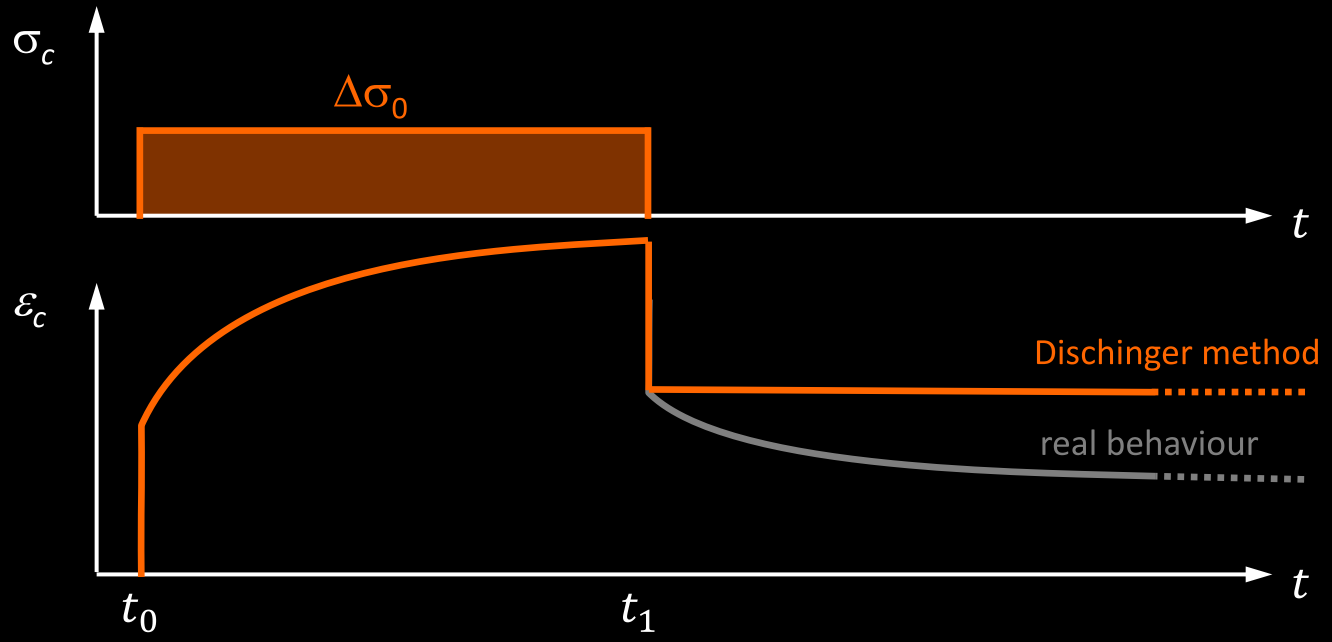


Long-term effects

Approaches for the calculation of creep and shrinkage problems

Unit creep curve method (Dischinger method)

- Same creep curve for all loads, shifted along the ordinate (i.e. vertically)
- Advantage: Representation in recursion formulae possible
- Unrealistic: underestimates the creep of old concrete
- Unrealistic: neglects viscoelastic behaviour (no reversible part)

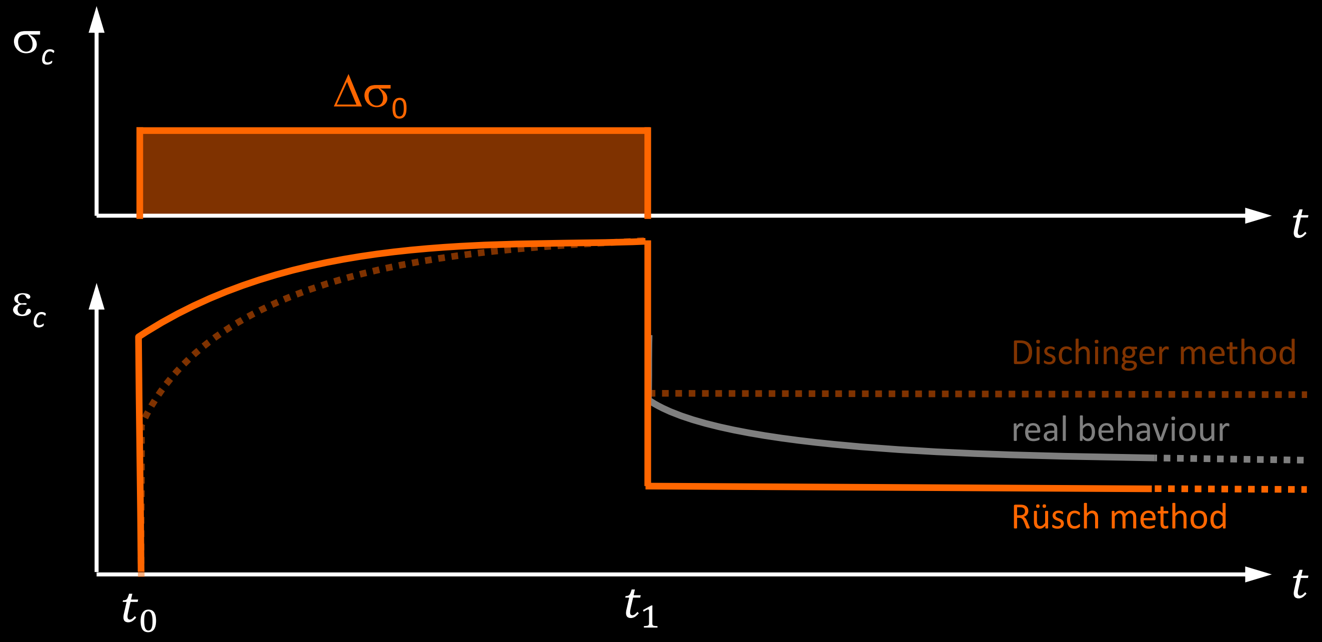


Long-term effects

Approaches for the calculation of creep and shrinkage problems

Rüsch method (improved Dischinger method)

- Basically the same assumptions as Dischinger method
- Superposition of the entire reversible part of creep deformations (neglected in the Dischinger method) with the elastic elongation
- Reasonably realistic, since the reversible portion of creep deformations occurs relatively quickly



Long-term effects

Approaches for the calculation of creep and shrinkage problems

Creep step method

- The stress history is only known in advance in simple cases (this was assumed in the previous considerations). In general, it depends on the creep behaviour. The solution therefore usually requires an iterative or step-by-step approach.
- Based on the Dischinger method a differential equation for creep behaviour can be formulated (also possible with the Rüsch method). For numerical solutions, the creep step method can be used, which is based on a subdivision of the load history into time intervals or into "creep steps" (subdivision of the creep coefficient $\varphi(t=\infty, t_0)$ in equal creep intervals $\Delta\varphi$, usually more appropriate).
- Linearisation of the creep and stress function per interval results in the increase of creep deformation in the time interval. $\Delta t_i = t_i - t_{i-1}$ (note that since Dischinger's Method is used, the reversible part of creep is not accounted for):

$$\Delta\varepsilon_{cc,i} = \frac{\sigma_{i-1}}{E_{c0}}\Delta\varphi_i + \frac{\Delta\sigma_i}{E_{c0}}\frac{\Delta\varphi_i}{2} = \frac{\sigma_{i-1} + \Delta\sigma_i/2}{E_{c0}}\Delta\varphi_i;$$

$\Delta\varphi_i = \varphi_i - \varphi_{i-1}$: Change of the creep function during Δt_i

$\Delta\sigma_i = \sigma_i - \sigma_{i-1}$: Change of the concrete stress during Δt_i

- Total strain increase in time interval $\Delta t_i = t_i - t_{i-1}$:

$$\Delta\varepsilon_{c,i} = \frac{\Delta\sigma_i}{E_{c0}} + \Delta\varepsilon_{cc,i} + \Delta\varepsilon_{cs,i} = \frac{\Delta\sigma_i}{E_{c0}} + \frac{\sigma_{i-1}}{E_{c0}}\Delta\varphi_i + \frac{1}{2}\frac{\Delta\sigma_i}{E_{c0}}\Delta\varphi_i + \Delta\varepsilon_{cs,i} = \frac{\Delta\sigma_i}{E_{c0}} + \frac{\sigma_{i-1} + \Delta\sigma_i/2}{E_{c0}}\Delta\varphi_i + \Delta\varepsilon_{cs,i}$$