2 In-plane loading – membrane elements

2.5 Compatibility and deformation capacity

Learning objectives

Within this chapter, the students are able to:

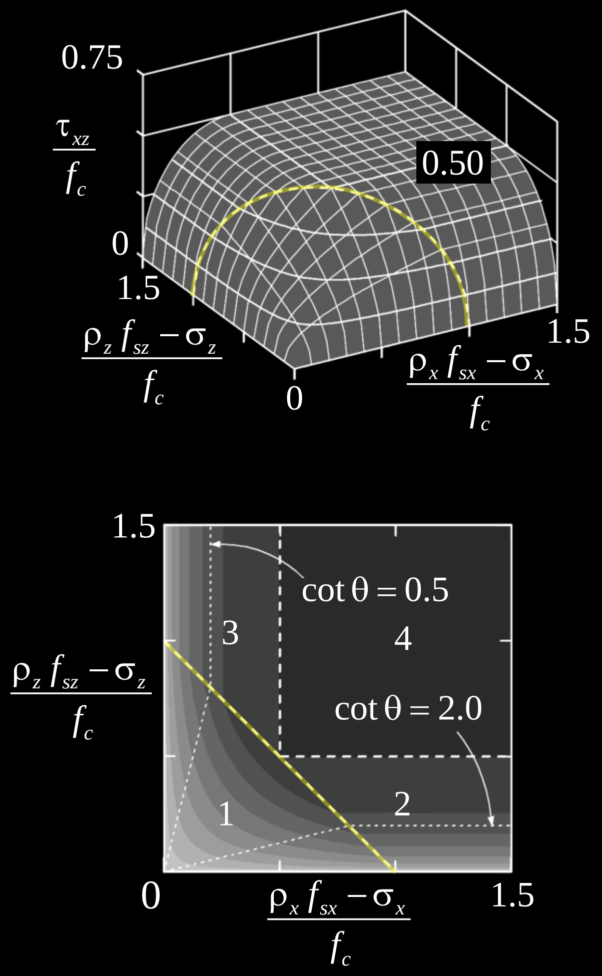
- describe how using an effective compressive strength dependent on the transverse strain state modifies the boundaries of the membrane yield conditions.
- discuss the differences and similitudes between various compression field models which can be used to investigate the load-deformation behaviour of reinforced concrete membrane elements.
- formulate the main assumptions of the Cracked Membrane Model with stress-free cracks, including how to model tension stiffening for bidirectional reinforcement using the Tension Chord Model.

2 In-plane loading – membrane elements

2.5 Compatibility and deformation capacity

A) Influence of strains on the compressive strength and thus on the yield conditions

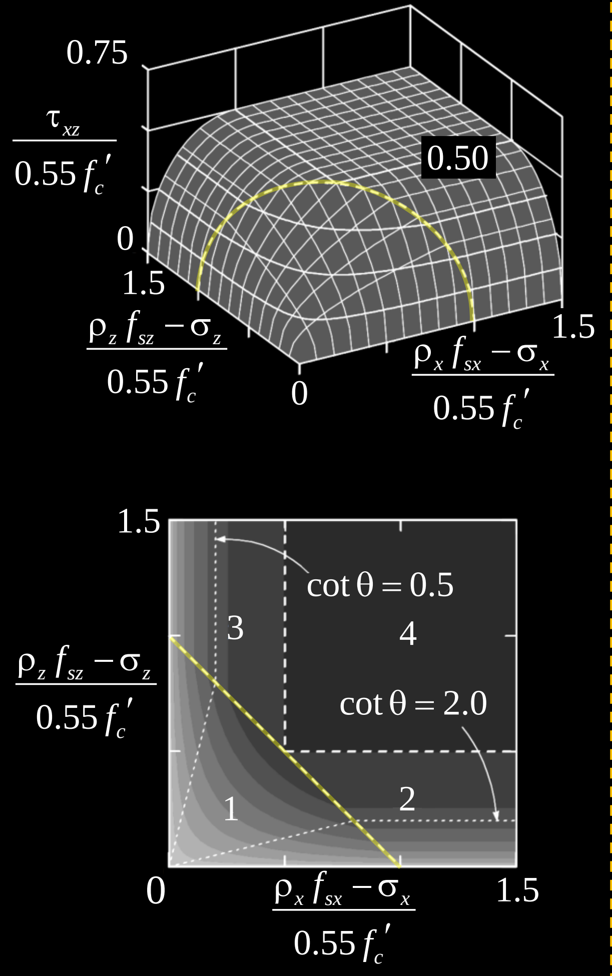
Membrane elements – Effective compressive strength



Constant concrete compressive strength
(f_c independent of ε_1)

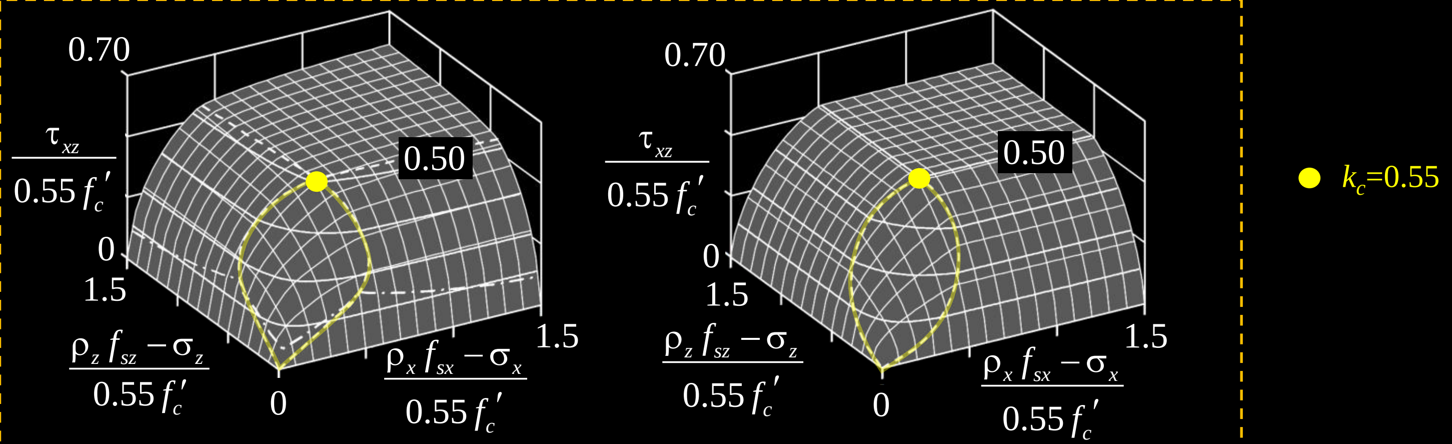
Membrane elements – Effective compressive strength

f_c=const. / Particularised for $f_c = k_c f_c' = 0.55 f_c'$

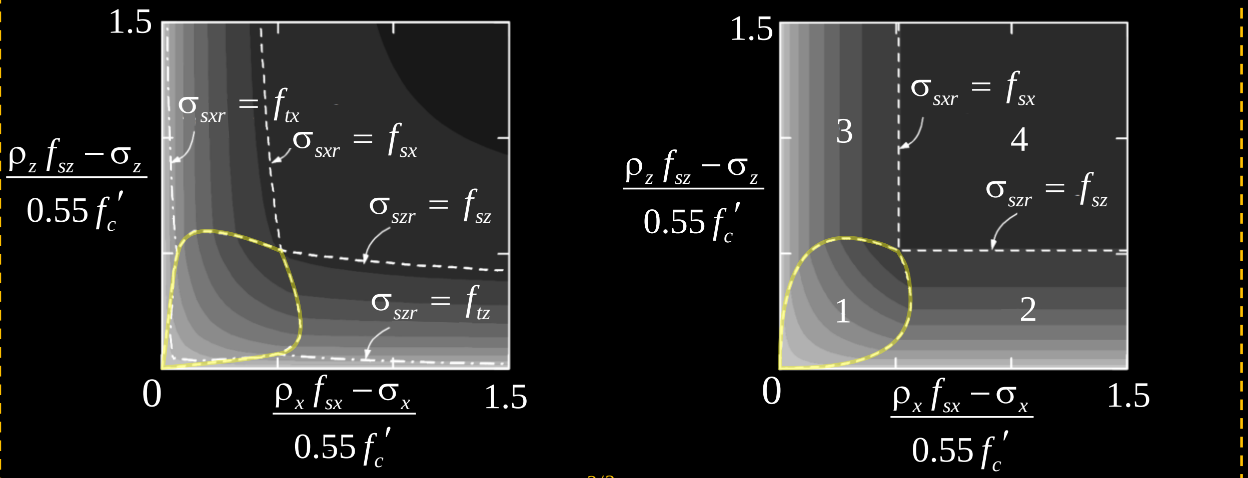


Constant concrete compressive strength (f_c independent of ε_1)

$f_c=f(\varepsilon_1)$ / Particularised for $f_c' = 30$ MPa



Boundary of Regime 1: concrete crushes, stronger reinforcement at onset of yielding → ε_1 can be approximated



"Exact" calculation with CMM approach [MPa]: $f_c = k_c f_c' = \dfrac{(f_c')^{2/3}}{0.4 + 30 \cdot \varepsilon_1}$

Approximation with simplified ε_1 along boundary Y1 (see next slide)

Membrane elements – Effective compressive strength

Influence on yield conditions

The yield surface can be modified by taking into account the dependence of the concrete compressive strength on the transverse strains.

→ Area of Regime 1 is reduced (affected: zones with very flat / steep inclinations)

→ Calculation with Cracked Membrane Model (CMM, middle graph) is tedious

→ Approximate solution (bottom graphs):

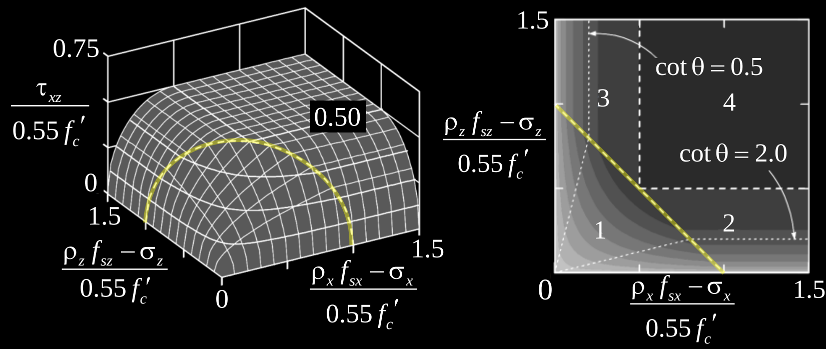
assuming: $$k_c f_c' = \frac{(f_c')^{2/3}}{0.4 + 30 \cdot \varepsilon_1}$$ (1998):

$$Y_1: \quad \tau_{xz}^2 = (\rho_x f_{sx} - \sigma_x)(\rho_z f_{sz} - \sigma_z) \quad \text{(unchanged)}$$

$$Y_2: \quad \tau_{xz}^2 = (\rho_z f_{sz} - \sigma_z)^2 \left\{ \sqrt{2.0 + \frac{25}{3} \frac{(f_c')^{2/3}}{(\rho_z f_{sz} - \sigma_z)}} - \frac{29}{12} \right\}$$

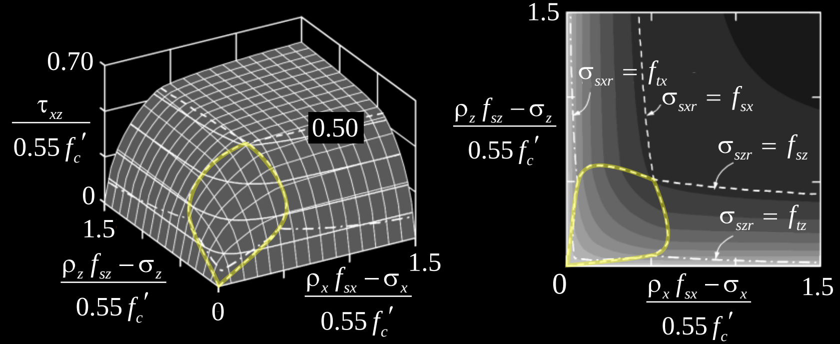
$$Y_3: \quad \tau_{xz}^2 = (\rho_x f_{sdx} - \sigma_x)^2 \left\{ \sqrt{2.0 + \frac{25}{3} \frac{(f_c')^{2/3}}{(\rho_x f_{sx} - \sigma_x)}} - \frac{29}{12} \right\}$$

$$Y_4: \quad \tau_{xz}^2 = \left\{ \frac{25}{29} (f_c')^{2/3} \right\}^2$$



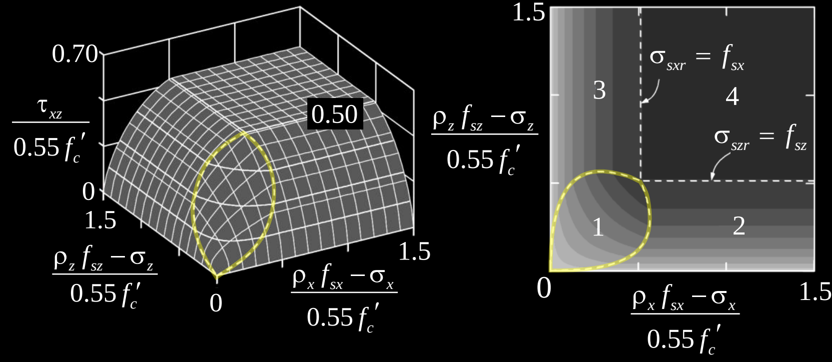
$f_c = 0.55 f_c'$

$f_c' = 30$ MPa



$f_c = f(\varepsilon_1)$ **CMM**

Boundary Regime 1: concrete crushes, stronger reinforcement at onset of yielding → ε_1 can be determined



$f_c = f(\varepsilon_1)$ **ε_1 simpl.**

Membrane elements – Effective compressive strength

Influence on yield conditions

The yield surface can be modified by taking into account the dependence of the concrete compressive strength on the transverse strains.

$\rightarrow$ Area of Regime 1 is reduced (affected: zones with very flat / steep inclinations)

$\rightarrow$ Calculation with Cracked Membrane Model (CMM, middle graph) is tedious

$\rightarrow$ Approximate solution (bottom graphs):

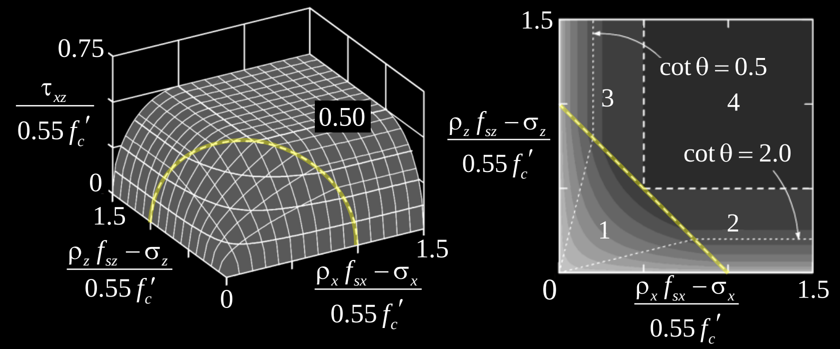
according to SIA: (2013), $k_c f_c = \dfrac{f_c}{1.2 + 55 \cdot \varepsilon_1}$:

$$Y_1: \quad \tau_{xz}^2 = (\rho_x f_{sdx} - \sigma_x)(\rho_z f_{sdz} - \sigma_z) \quad \text{(unchanged)}$$

$$Y_2: \quad \tau_{xz}^2 = (\rho_z f_{sdz} - \sigma_z)^2 \left\{ \sqrt{\frac{135}{22} + \frac{50}{11} \frac{f_c}{(\rho_z f_{sdz} - \sigma_z)}} - \frac{73}{21} \right\}$$

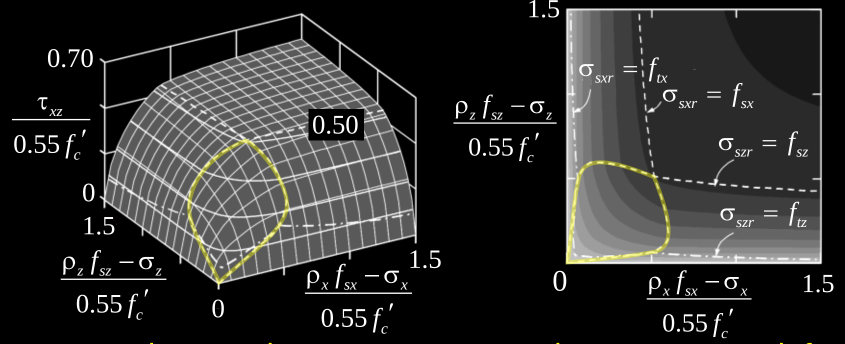
$$Y_3: \quad \tau_{xz}^2 = (\rho_x f_{sdx} - \sigma_x)^2 \left\{ \sqrt{\frac{135}{22} + \frac{50}{11} \frac{f_c}{(\rho_x f_{sdx} - \sigma_x)}} - \frac{73}{21} \right\}$$

$$Y_4: \quad \tau_{xz}^2 = \left\{ \frac{16}{49} f_c \right\}^2 \quad \text{(d.h. } \tau_{xz} = 0.327 \cdot f_c \approx \frac{0.65 \cdot f_c}{2}\text{)}$$



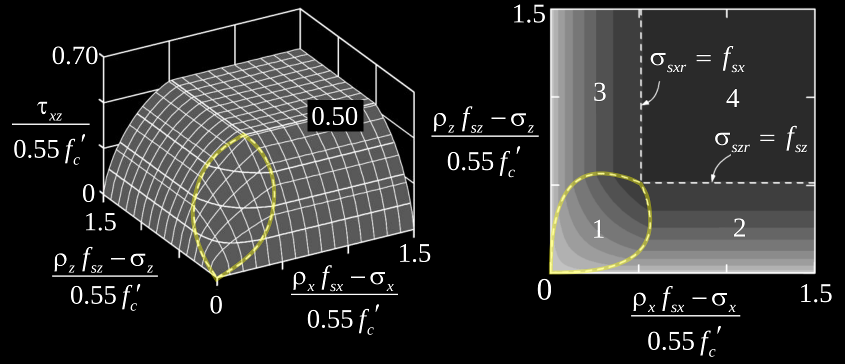
$f_c = 0.55 f_c'$

$f_c' = 30$ MPa



$f_c = f(\varepsilon_1)$
CMM

Boundary Regime 1: concrete crushes, stronger reinforcement at onset of yielding $\rightarrow \varepsilon_1$ can be determined



$f_c = f(\varepsilon_1)$
ε_1 **simpl.**

2 In-plane loading – membrane elements

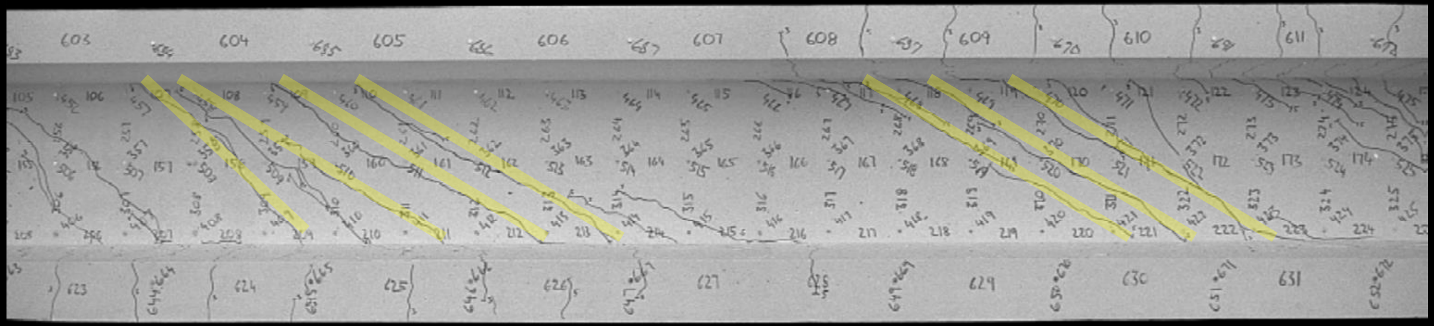
2.5 Compatibility and deformation capacity

B) Load-deformation behaviour of membranes

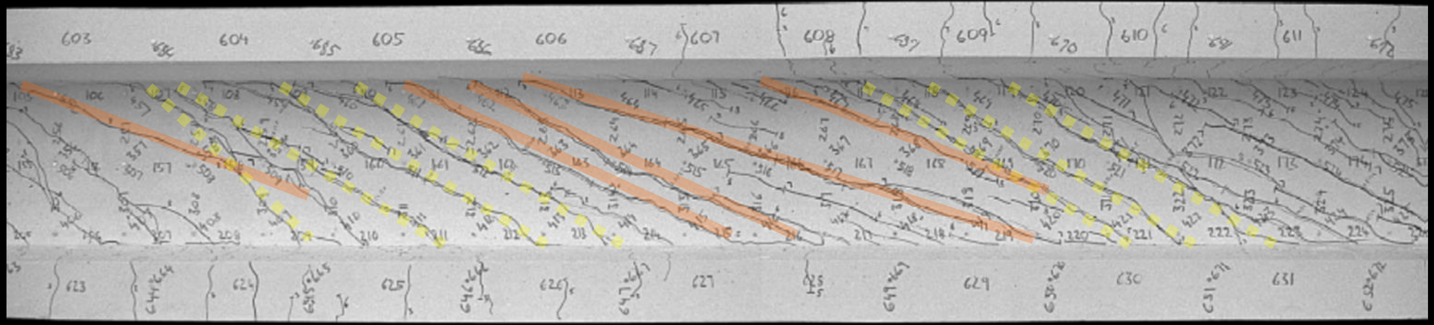
Membrane elements - Load-deformation behaviour

General

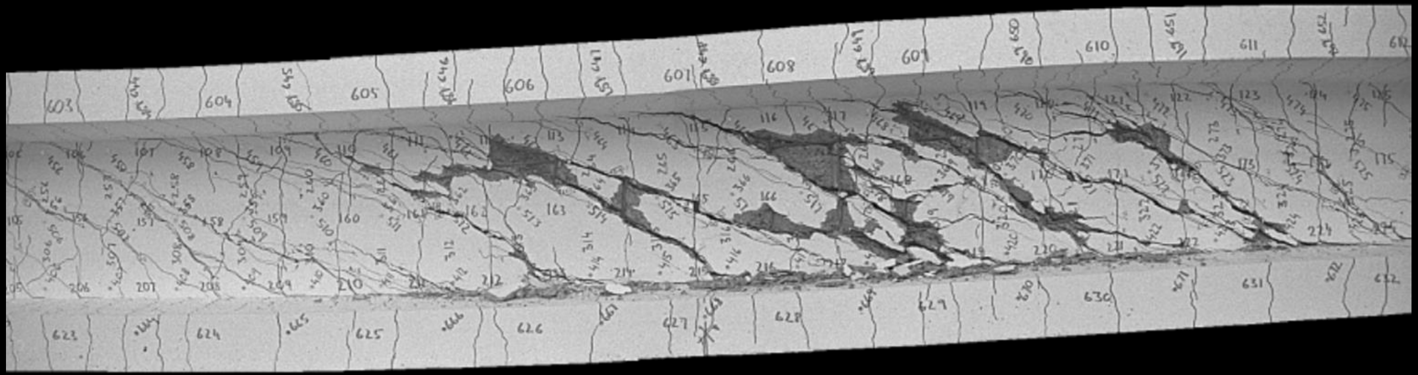
Experiment VN2
V = 360 kN
$\alpha_r \approx 30°$



Experiment VN2
V = 545 kN
$\alpha_r \approx 17…25°$



Experiment VN2
V = 548 kN
(failure)

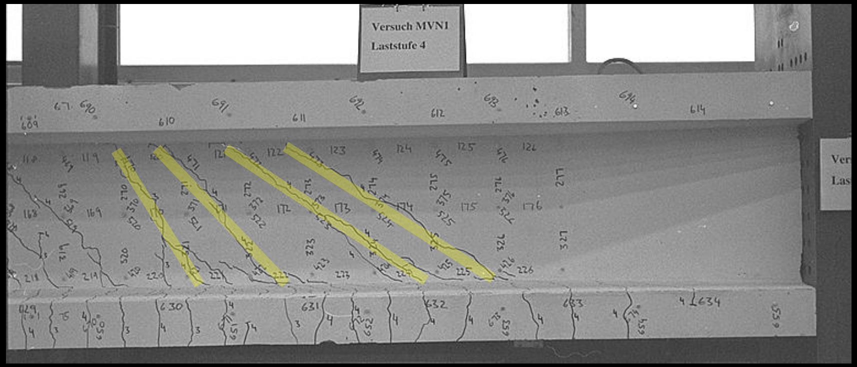


Membrane elements - Load-deformation behaviour

General

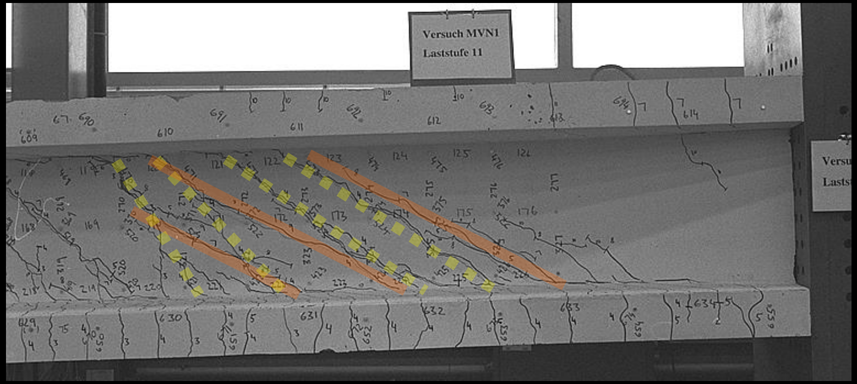
Experiment MVN1
V = 210 kN

$\alpha_r \approx$ 35...55°



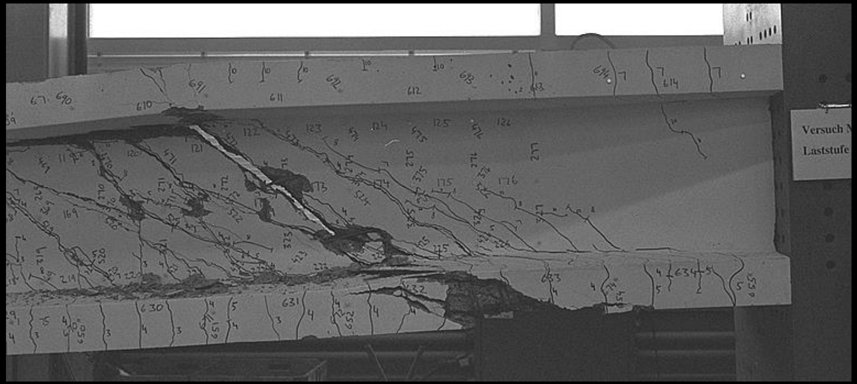
Experiment MVN1
V = 510 kN

$\alpha_r \approx$ 25°



Experiment MVN1
V = 540 kN

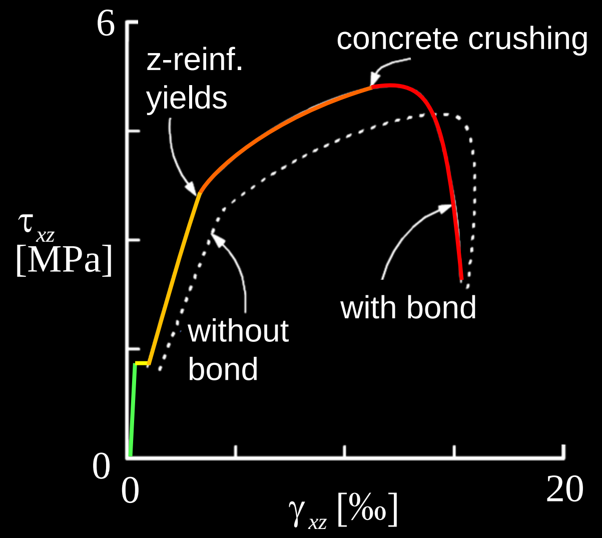
(failure)

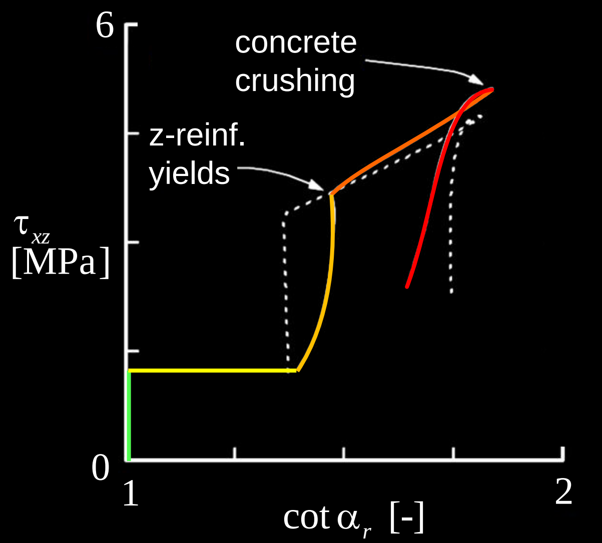


Membrane elements - Load-deformation behaviour

Reinforced concrete membrane element under monotonous load increase

1. Uncracked behaviour: Like homogeneous concrete membrane element (slight differences due to restraint shrinkage etc.)
2. Initial cracking approximately perpendicular to the principal tensile stress direction
3. Crack formation → Redistribution of internal forces → Change of principal stress directions immediately after crack formation
4. Cracked-elastic behaviour: Principal stress directions ± constant as long as both reinforcements remain elastic
5. Yielding of a reinforcement
   → Decrease in stiffness → Further redistribution of internal forces
   → New cracks (closer to the direction of the non-yielding reinforcement)
6. Failure due to crushing of the concrete or yielding of the other reinforcement (possibly reinforcement ruptures or aggregate interlock fails)

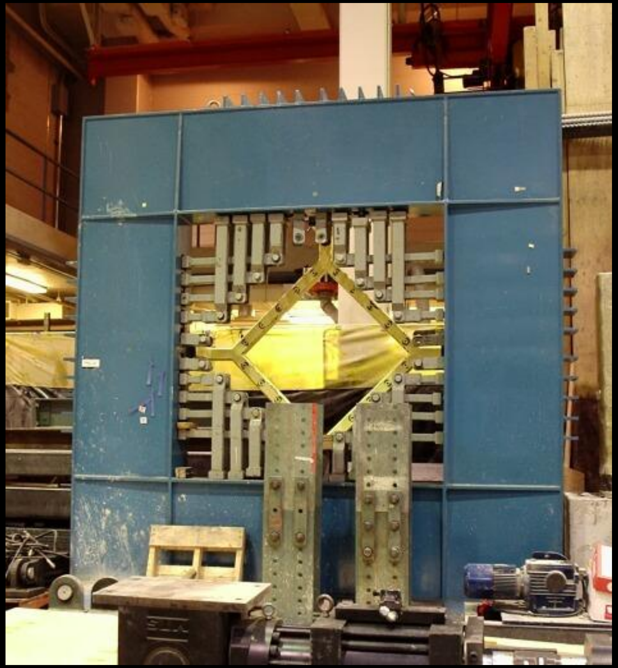




Membrane elements - Load-deformation behaviour

Test facilities for uniformly stressed elements

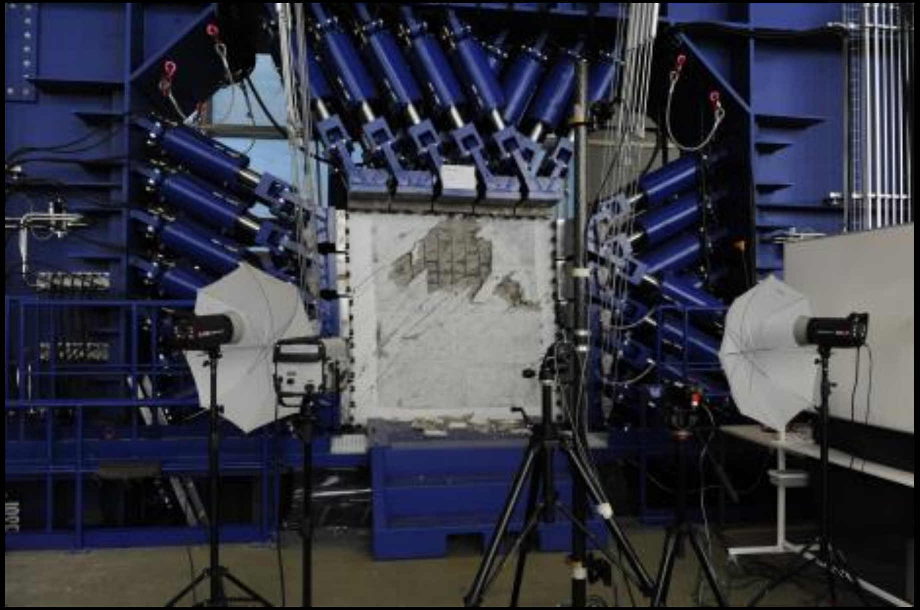
Shear Panel Tester
University of Toronto 1979



Shell Element Tester
University of Toronto 1984 / 2009

Large Universal Shell Element Tester
ETH Zürich 2017

Large Universal Shell Element Tester LUSET, ETH Zurich (2017)



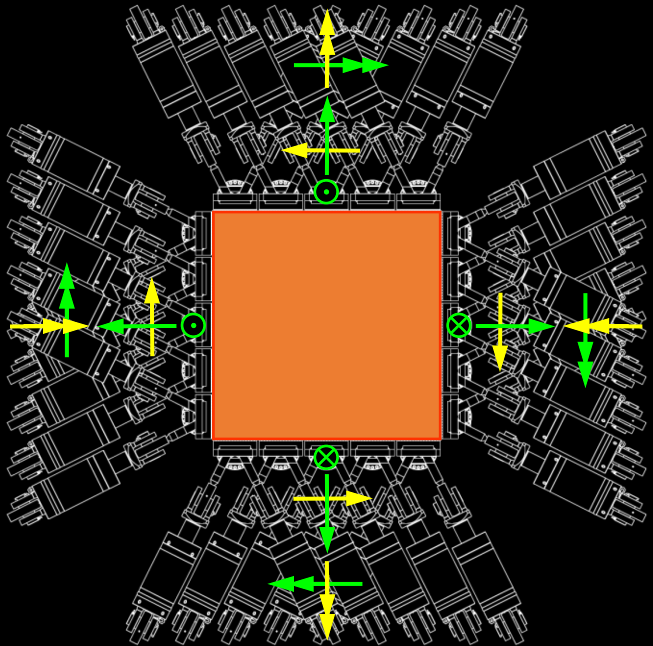
General loading (8 stress resultants)

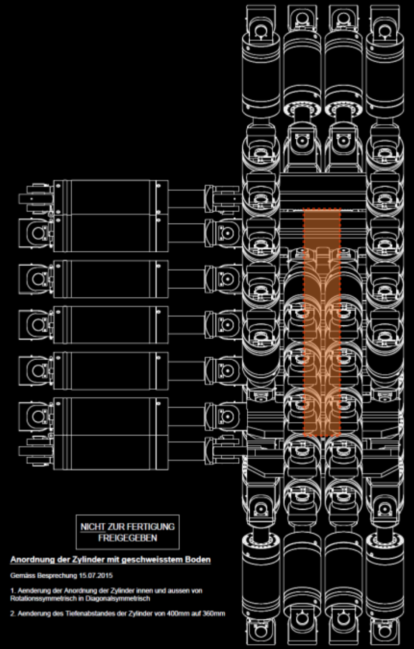
Applied loads in-plane and out-of-plane of general direction, i.e. perpendicular and parallel to element edge

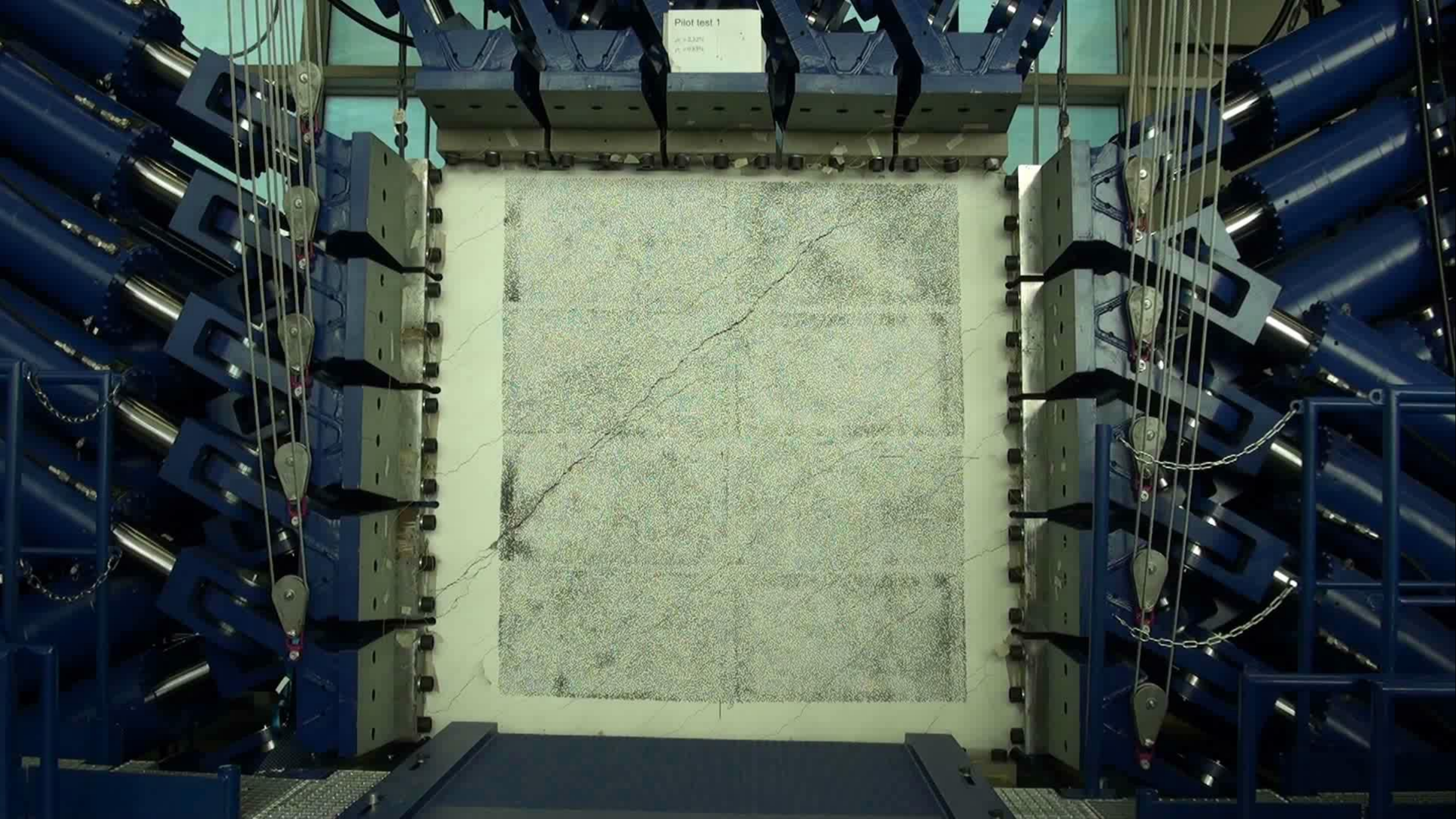
→ principal direction of applied loads variable

→ reinforcing bars parallel to element edges

Element size 2,000-2,000-350 mm






Pilot test 1

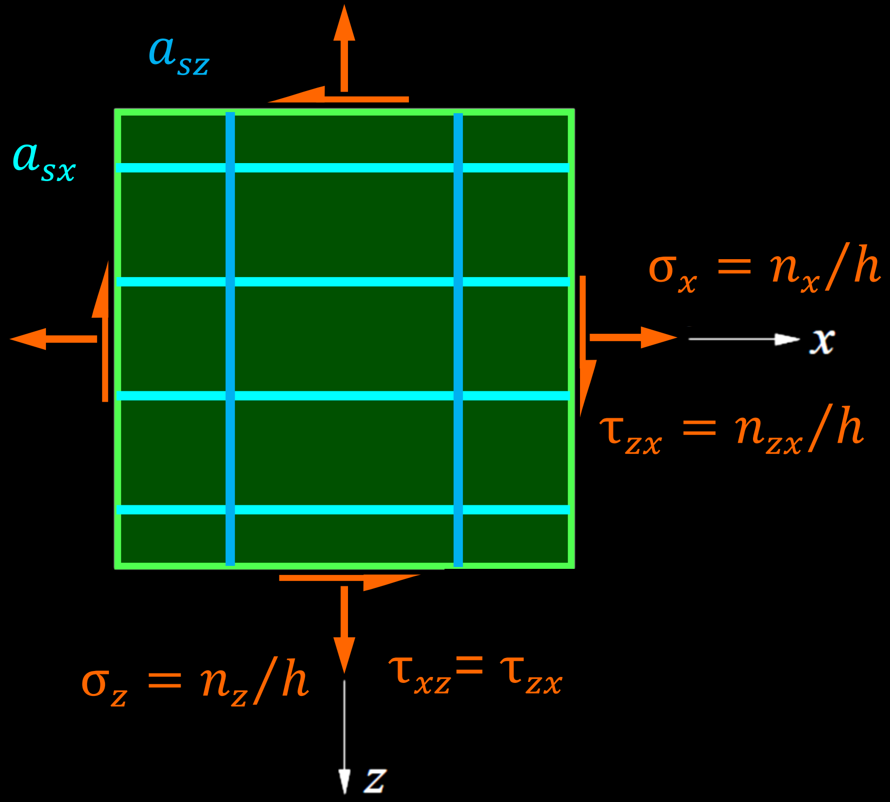
2 In-plane loading – membrane elements

2.5 Compatibility and deformation capacity

C) Compression field approaches

Membrane elements - Load-deformation behaviour

External loads are in equilibrium with reinforced concrete = concrete + reinforcing steel



Equilibrium of forces [kN/m]

$$n_x = n_{xc} + n_{xs} \qquad = n_{xc} + a_{sx}\sigma_{sx}$$
$$n_z = n_{zc} + n_{zs} \qquad = n_{xc} + a_{sz}\sigma_{sz}$$
$$n_{xz} = n_{xzc} + \cancel{n_{xzs}} \qquad = n_{xzc}$$

Orthogonal reinforcement (dowelling action is neglected)

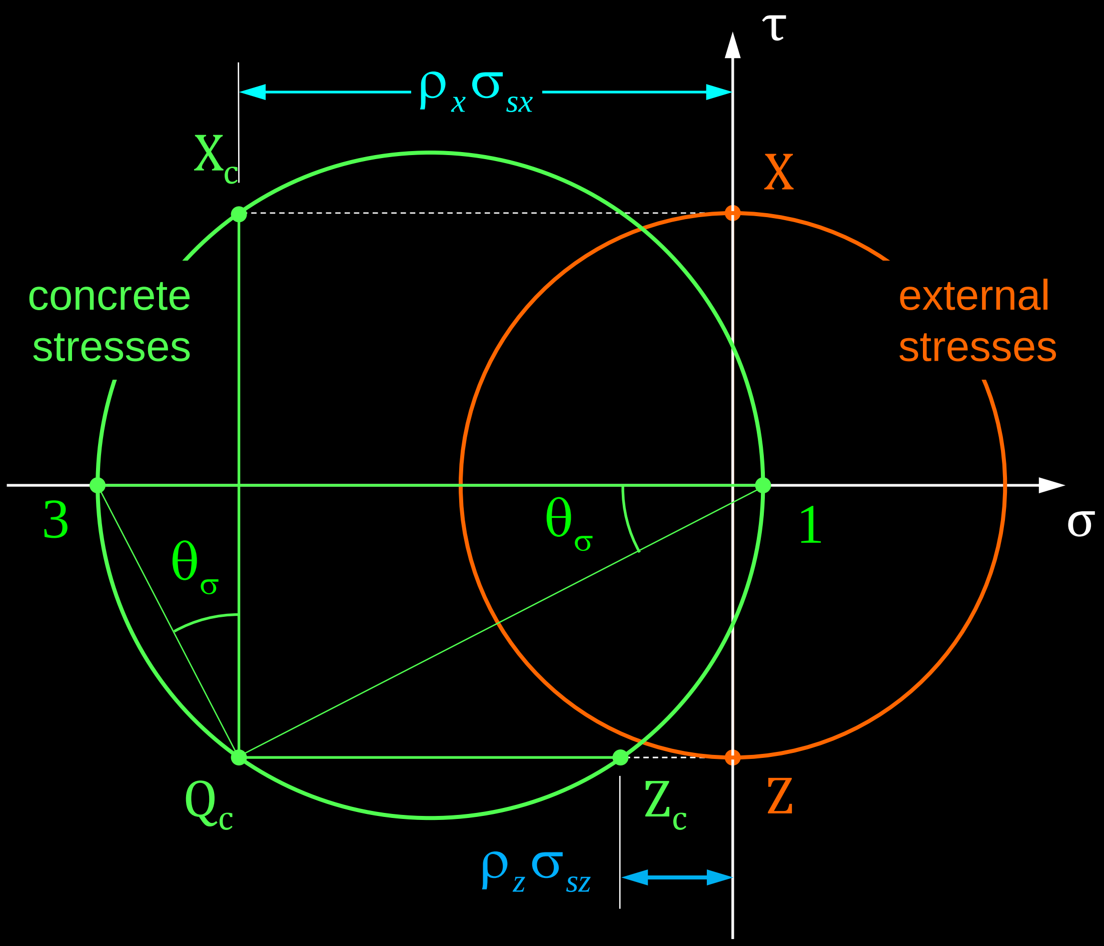
Equilibrium in equivalent stresses [MPa]

$$\sigma_x = \sigma_{xc} + \rho_x \sigma_{sx}$$
$$\sigma_z = \sigma_{zc} + \rho_z \sigma_{sz}$$
$$\tau_{xz} = \tau_{xzc}$$

(with $\rho_x\sigma_{sx}$, $\rho_z\sigma_{sz}$ = stresses in the reinforcement,
$\rho_x = a_{sx}/h$, $\rho_z = a_{sz}/h$)

Membrane elements - Load-deformation behaviour

External loads are in equilibrium with reinforced concrete = concrete + reinforcing steel



Equilibrium of forces [kN/m]

$$n_x = n_{xc} + n_{xs} \qquad = n_{xc} + a_{sx}\sigma_{sx}$$
$$n_z = n_{zc} + n_{zs} \qquad = n_{xc} + a_{sz}\sigma_{sz}$$
$$n_{xz} = n_{xzc} \qquad = n_{xzc}$$

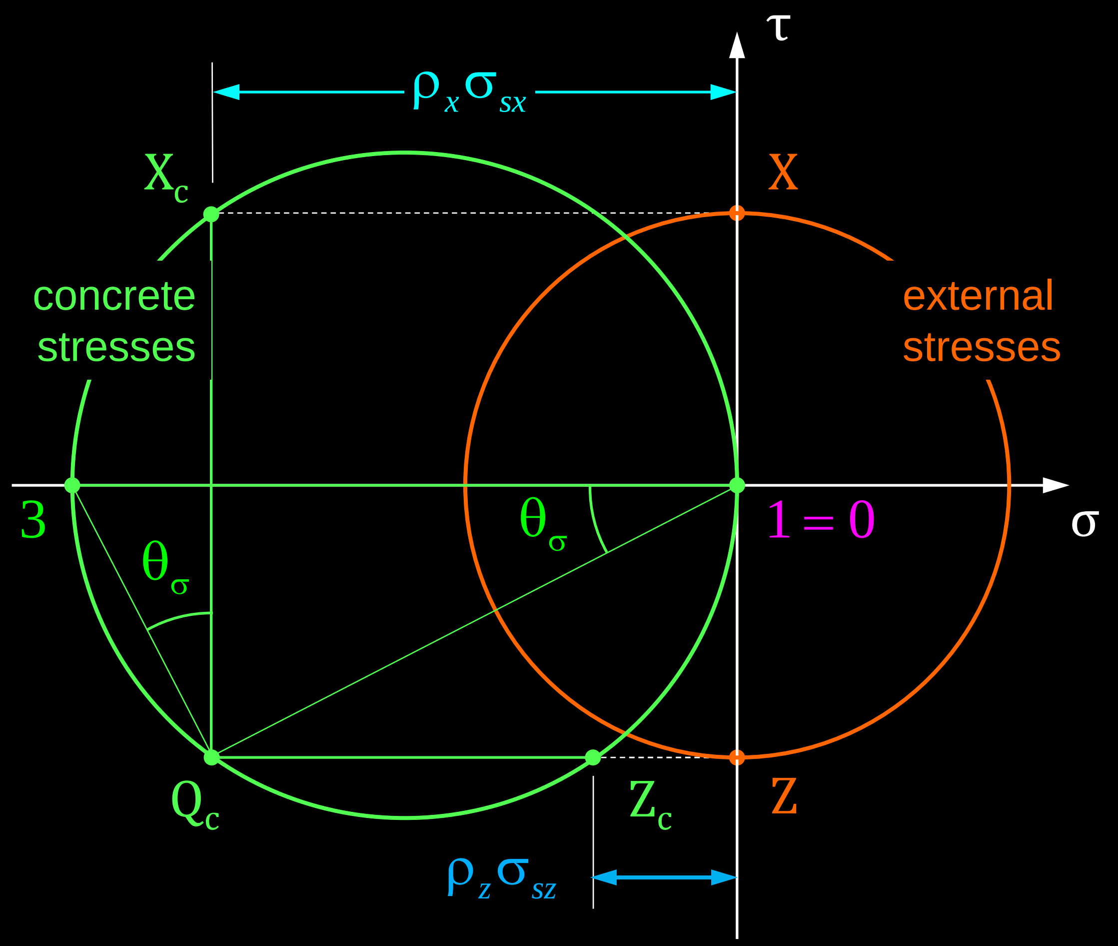
Equilibrium in equivalent stresses [MPa]

$$\sigma_x = \sigma_{c3}\cos^2\theta_\sigma + \sigma_{c1}\sin^2\theta_\sigma + \rho_x\sigma_{sx}$$
$$\sigma_z = \sigma_{c3}\sin^2\theta_\sigma + \sigma_{c1}\cos^2\theta_\sigma + \rho_z\sigma_{sz}$$
$$\tau_{xz} = \left(\sigma_{c1} - \sigma_{c3}\right)\sin\theta_\sigma\cos\theta_\sigma$$

(with $\rho_x\sigma_{sx}$, $\rho_z\sigma_{sz}$ = stresses in the reinforcement,
$\rho_x = a_{sx}/h$, $\rho_z = a_{sz}/h$)

Membrane elements - Load-deformation behaviour

External loads are in equilibrium with reinforced concrete = concrete + reinforcing steel



Equilibrium of forces [kN/m]

$$n_x = n_{xc} + n_{xs} \qquad = n_{xc} + a_{sx}\sigma_{sx}$$
$$n_z = n_{zc} + n_{zs} \qquad = n_{xc} + a_{sz}\sigma_{sz}$$
$$n_{xz} = n_{xzc} \qquad = n_{xzc}$$

Equilibrium in equivalent stresses [MPa]

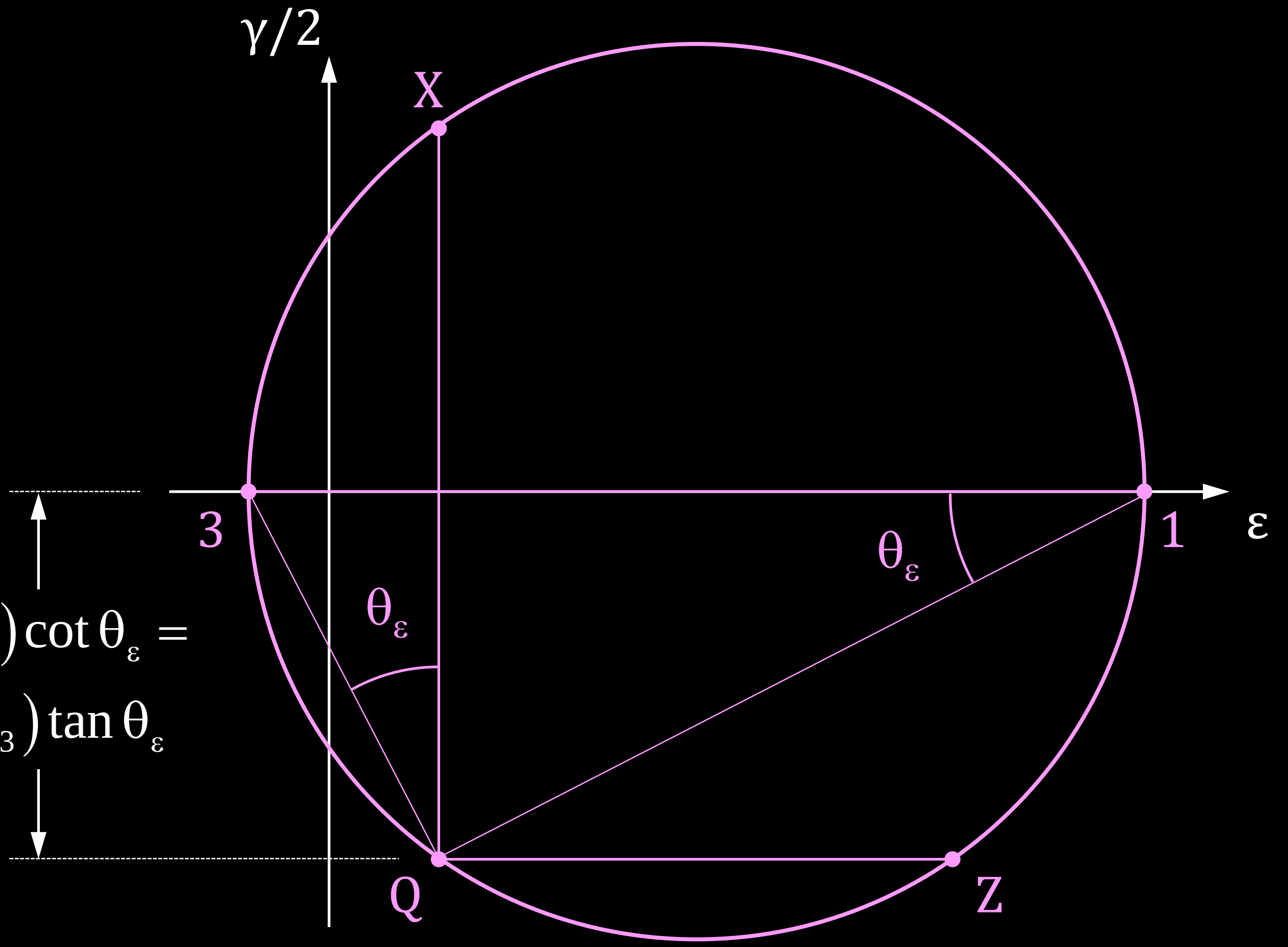
$$\sigma_x = \sigma_{c3}\cos^2\theta_\sigma \cancel{+\sigma_{c1}\sin^2\theta_\sigma} + \rho_x\sigma_{sx}$$
$$\sigma_z = \sigma_{c3}\sin^2\theta_\sigma \cancel{+\sigma_{c1}\cos^2\theta_\sigma} + \rho_z\sigma_{sz}$$
$$\tau_{xz} = (\cancel{\sigma_{c1}} - \sigma_{c3})\sin\theta_\sigma\cos\theta_\sigma$$

$\sigma_{c1} = 0$ (uniaxial compression in concrete, i.e. stress-free cracks with variable direction)

(with $\rho_x\sigma_{sx}$, $\rho_z\sigma_{sz}$ = stresses in the reinforcement, $\rho_x = a_{sx}/h$, $\rho_z = a_{sz}/h$)

Membrane elements - Load-deformation behaviour

Compatibility - Mohr's strain circle



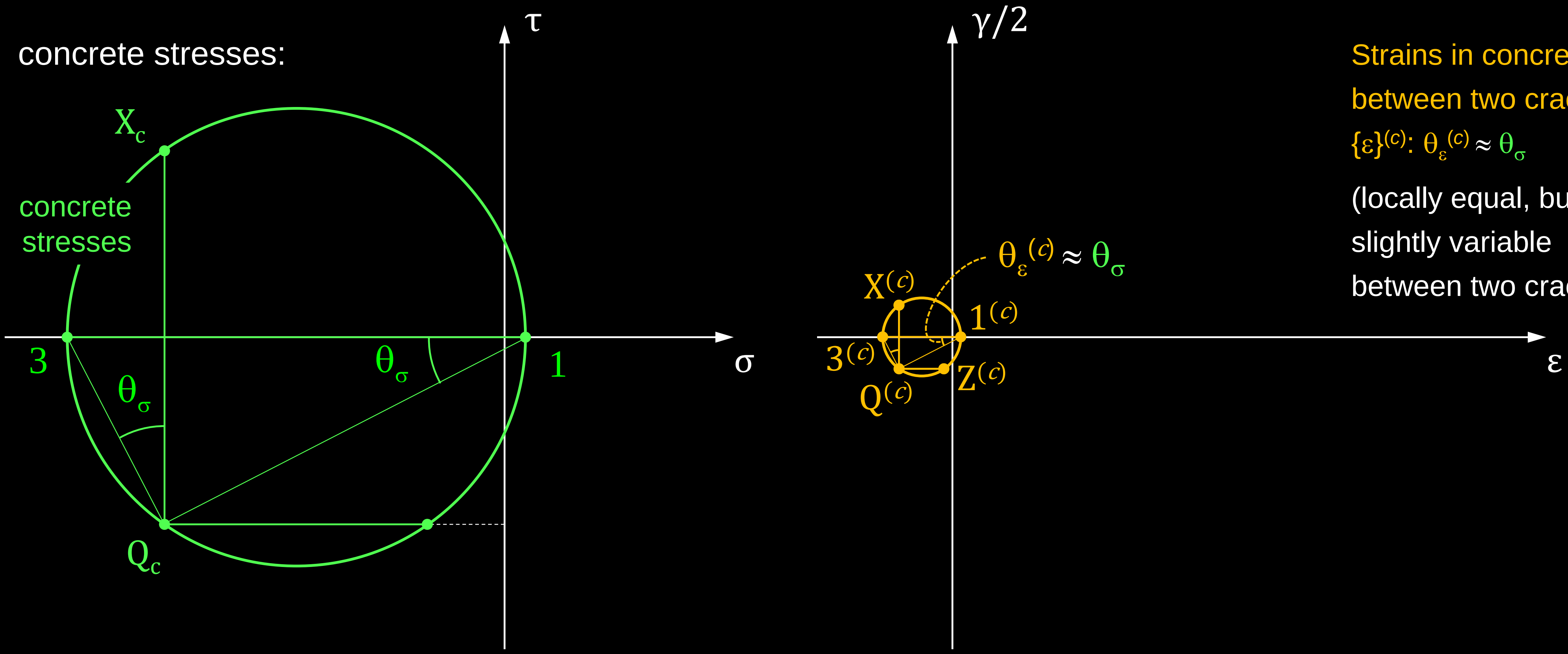
$$\left(\varepsilon_x - \varepsilon_3\right)\cot\theta_\varepsilon = \left(\varepsilon_z - \varepsilon_3\right)\tan\theta_\varepsilon$$

$$\cot^2\theta_\varepsilon = \frac{\varepsilon_z - \varepsilon_3}{\varepsilon_x - \varepsilon_3}$$

Strains in cracked membrane elements

Total strains {ε} = strains in concrete between cracks {ε}$^{(c)}$ + average strains due to crack kinematics {ε}$^{(r)}$

concrete stresses:



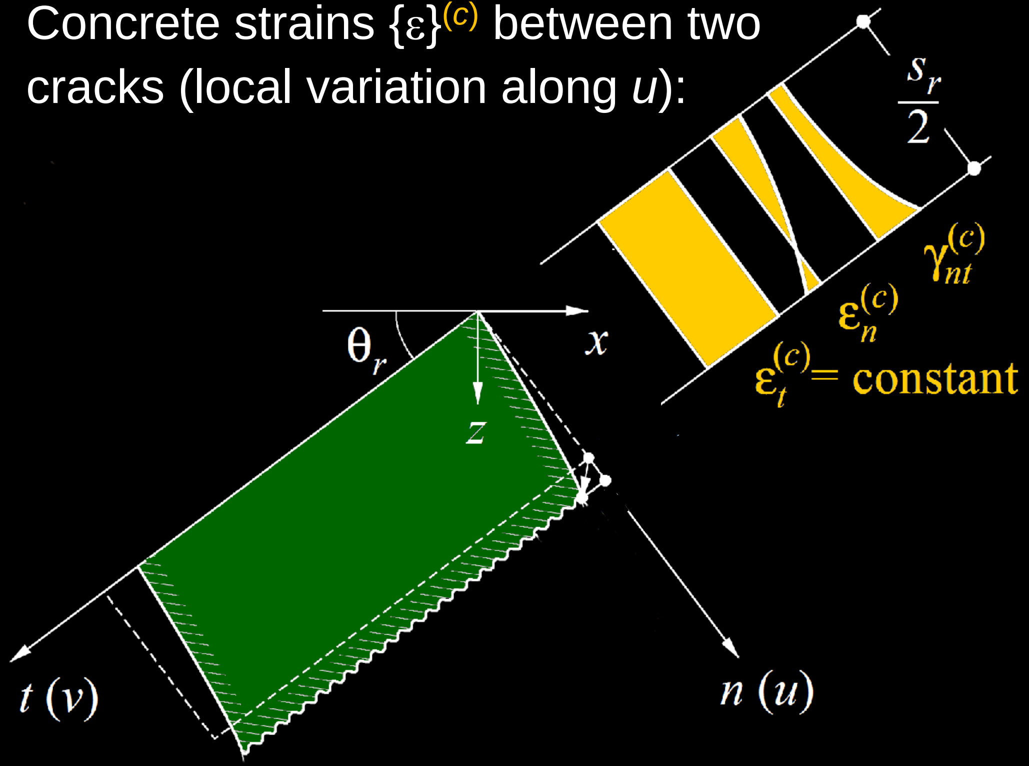
Strains in concrete between two cracks {ε}$^{(c)}$: $\theta_\varepsilon^{(c)} \approx \theta_\sigma$

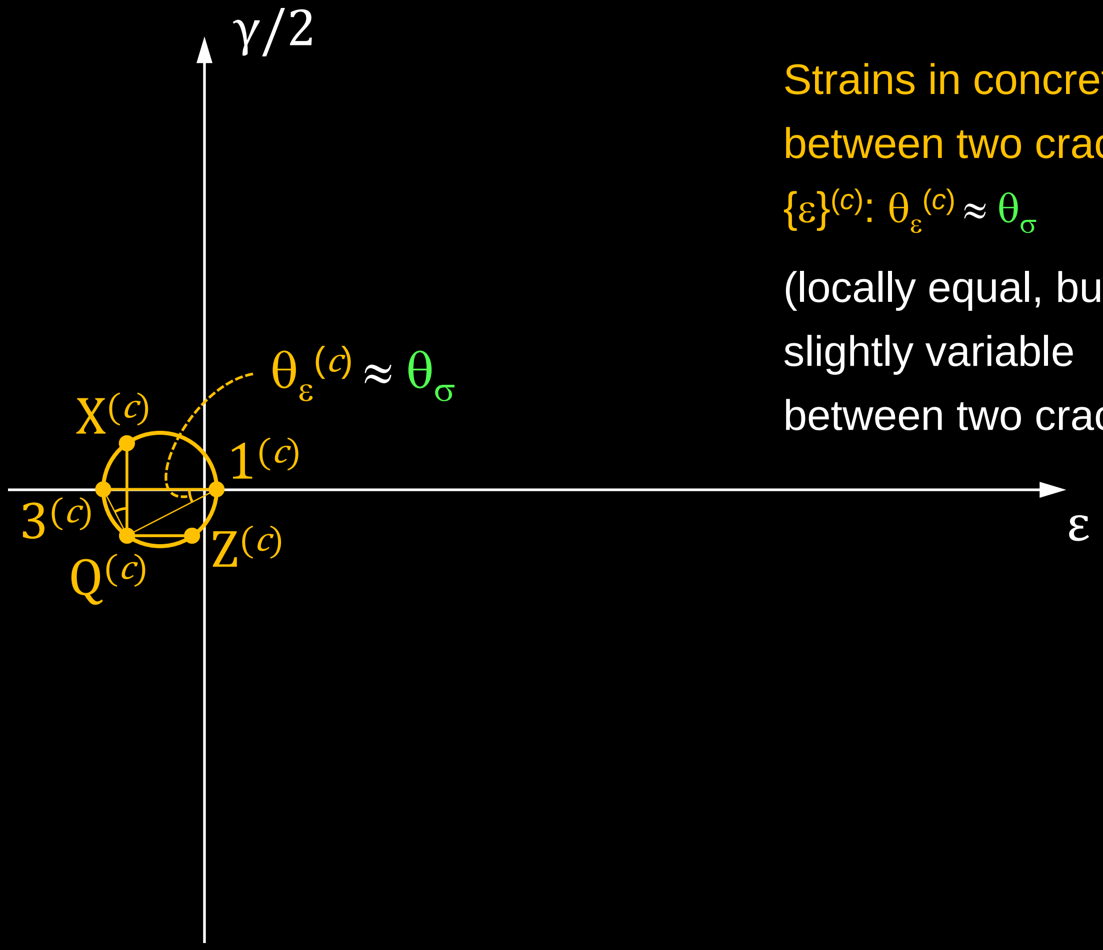
(locally equal, but θ_σ is slightly variable between two cracks)

Strains in cracked membrane elements

Total strains $\{\varepsilon\}$ = strains in concrete between cracks $\{\varepsilon\}^{(c)}$ + average strains due to crack kinematics $\{\varepsilon\}^{(r)}$

Concrete strains $\{\varepsilon\}^{(c)}$ between two cracks (local variation along *u*):





Strains in concrete between two cracks $\{\varepsilon\}^{(c)}$: $\theta_\varepsilon^{(c)} \approx \theta_\sigma$

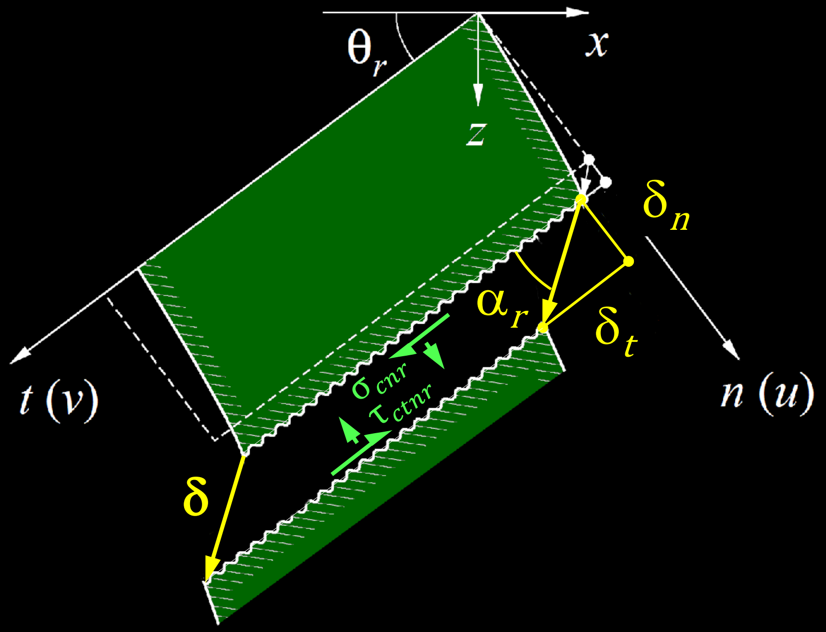
(locally equal, but θ_σ is slightly variable between two cracks)

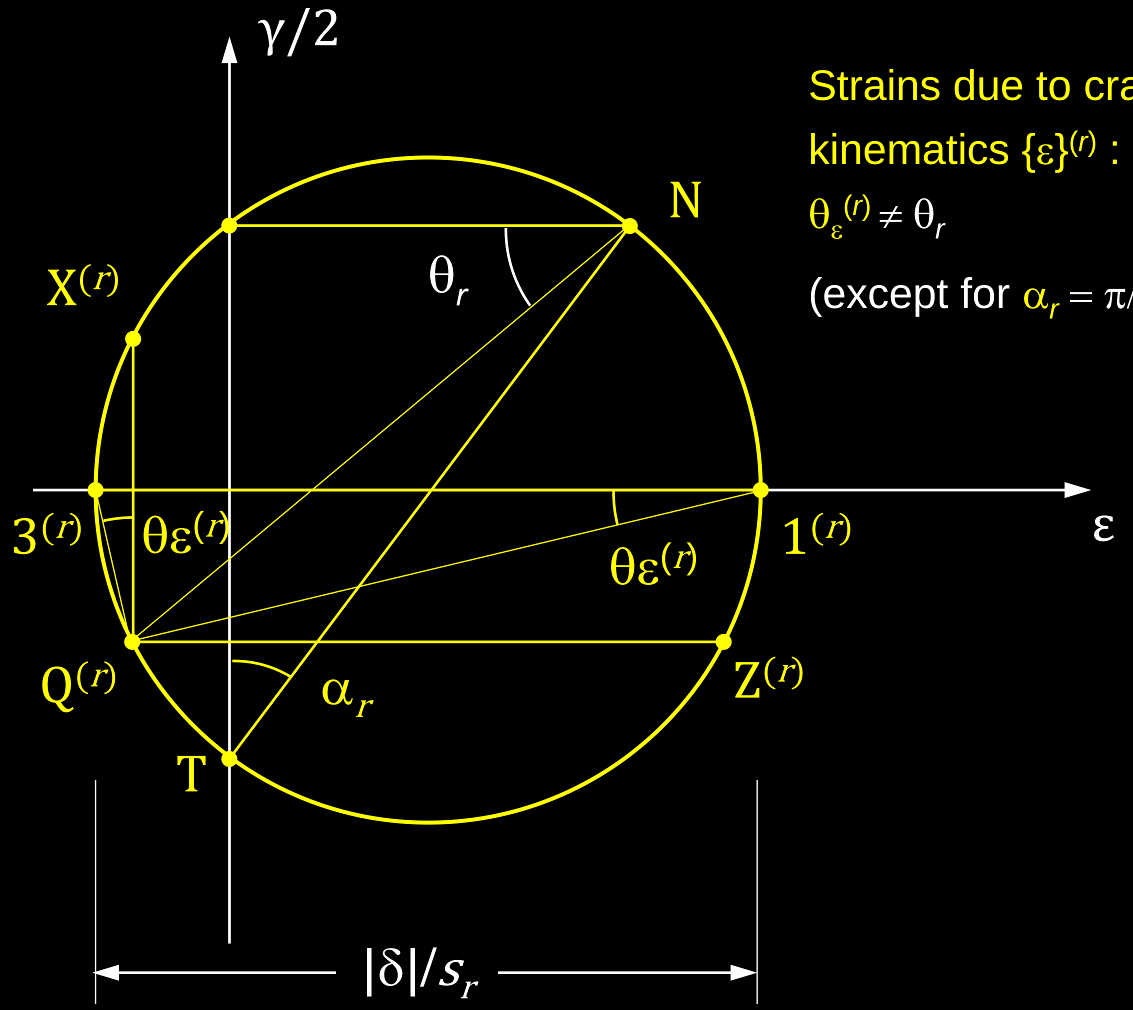
Strains in cracked membrane elements

Total strains $\{\varepsilon\}$ = strains in concrete between cracks $\{\varepsilon\}^{(c)}$ + average strains due to crack kinematics $\{\varepsilon\}^{(r)}$

Crack kinematics (parallel set of cracks):

s_r crack spacing
θ_r crack inclination
n, t coordinates $\perp$ and // to the crack





Strains due to crack kinematics $\{\varepsilon\}^{(r)}$:

$\theta_\varepsilon^{(r)} \neq \theta_r$

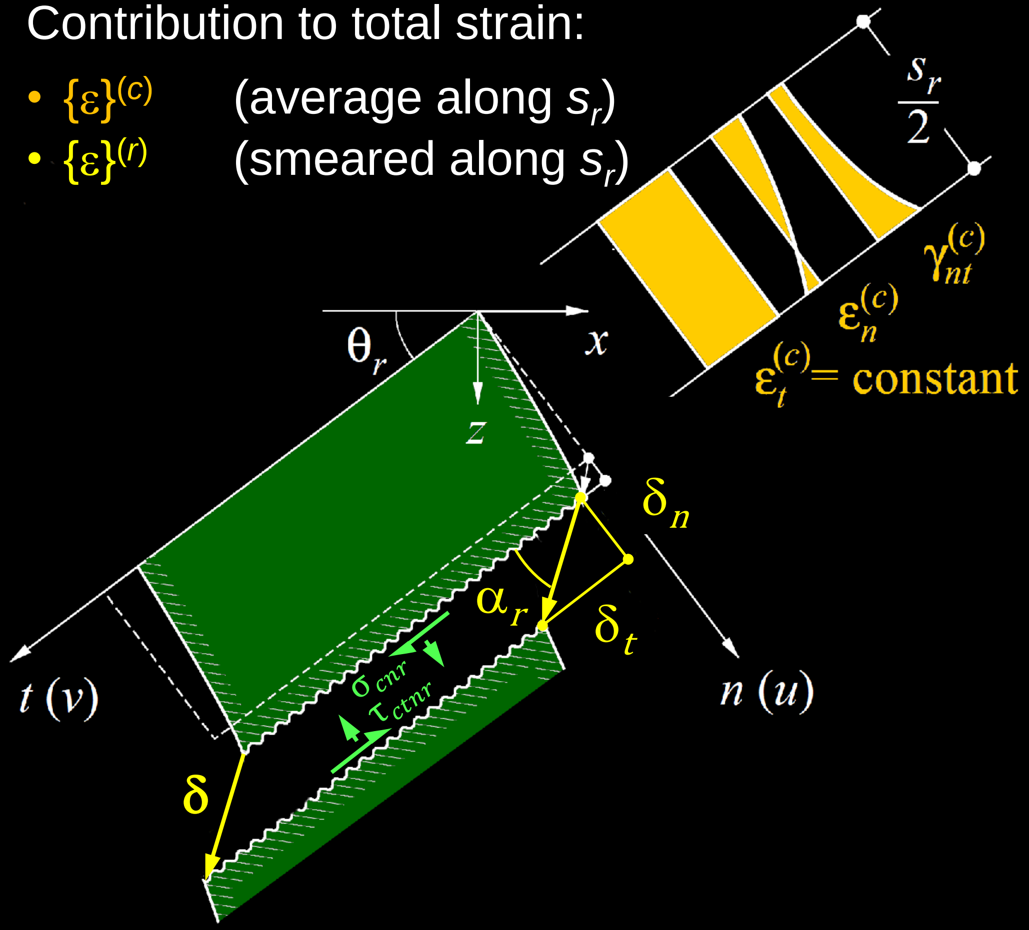
(except for $\alpha_r = \pi/2$)

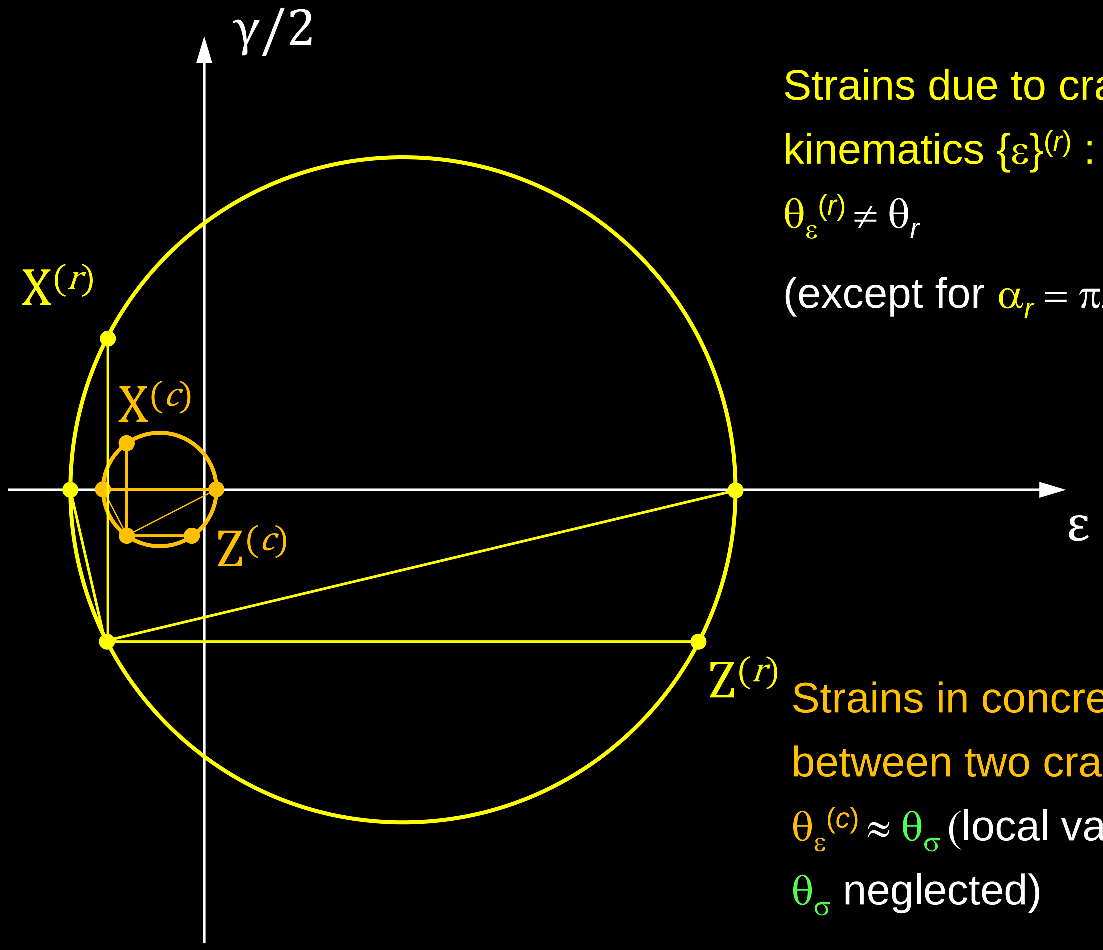
Strains in cracked membrane elements

Total strains $\{\varepsilon\}$ = strains in concrete between cracks $\{\varepsilon\}^{(c)}$ + average strains due to crack kinematics $\{\varepsilon\}^{(r)}$

Contribution to total strain:

- $\{\varepsilon\}^{(c)}$ (average along s_r)
- $\{\varepsilon\}^{(r)}$ (smeared along s_r)





Strains due to crack kinematics $\{\varepsilon\}^{(r)}$:

$\theta_\varepsilon^{(r)} \neq \theta_r$

(except for $\alpha_r = \pi/2$)

Strains in concrete between two cracks $\{\varepsilon\}^{(c)}$:

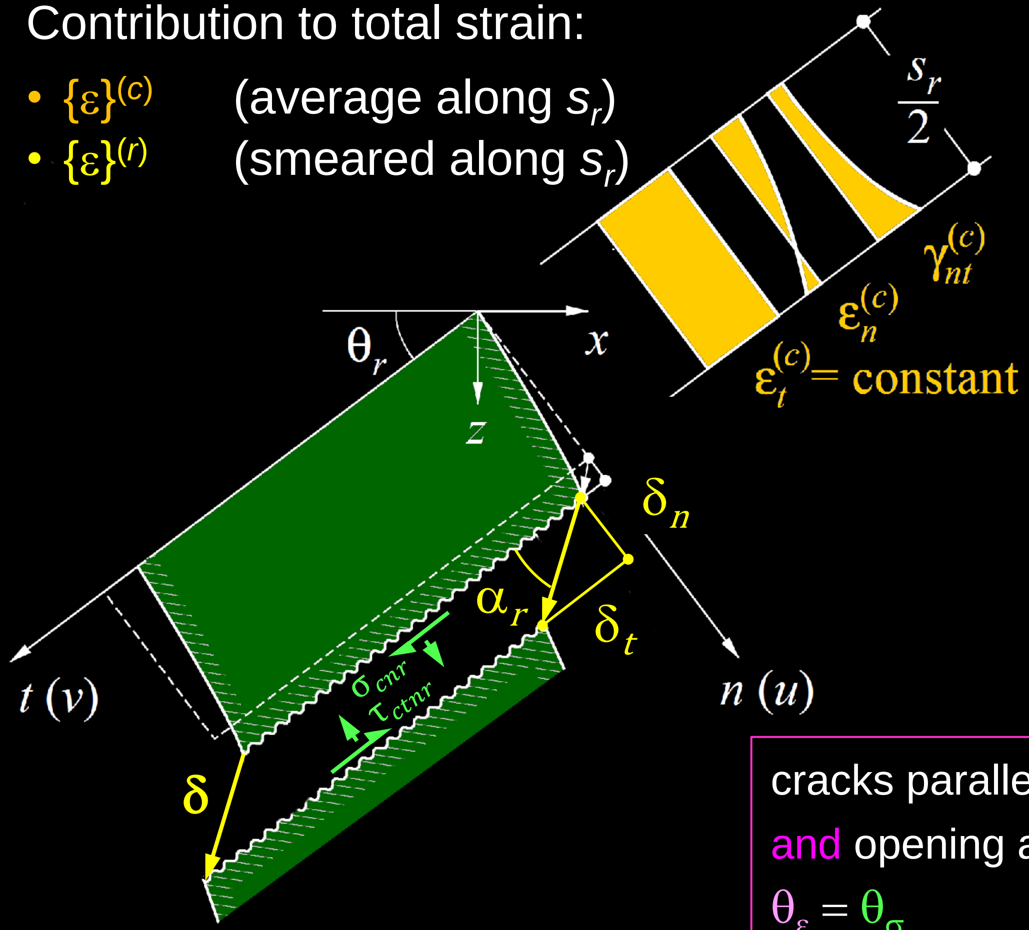
$\theta_\varepsilon^{(c)} \approx \theta_\sigma$ (local variation of θ_σ neglected)

Strains in cracked membrane elements

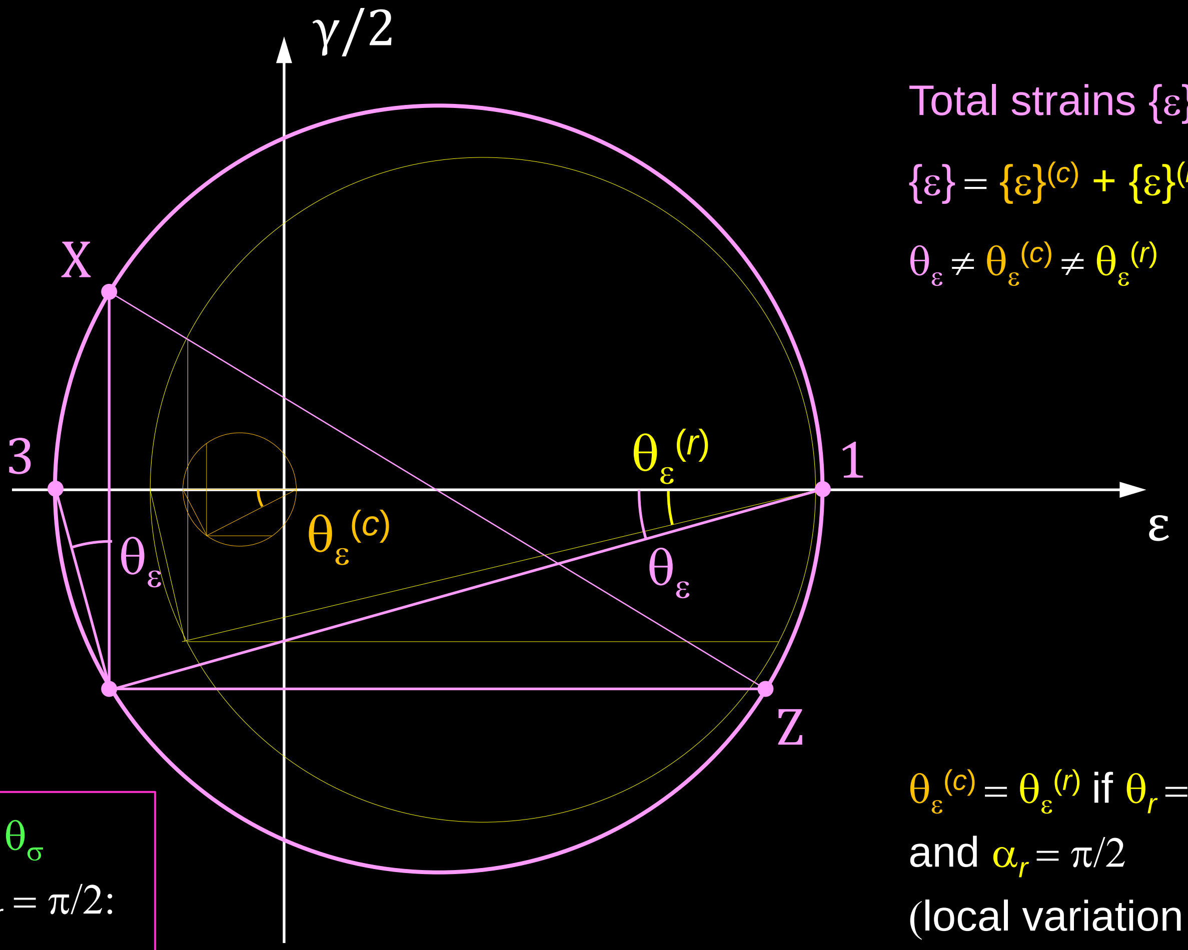
Total strains $\{\varepsilon\}$ = strains in concrete between cracks $\{\varepsilon\}^{(c)}$ + average strains due to crack kinematics $\{\varepsilon\}^{(r)}$

Contribution to total strain:

- $\{\varepsilon\}^{(c)}$ (average along s_r)
- $\{\varepsilon\}^{(r)}$ (smeared along s_r)



cracks parallel to θ_σ
and opening at $\alpha_r = \pi/2$:
$\theta_\varepsilon = \theta_\sigma$



Total strains $\{\varepsilon\}$:

$\{\varepsilon\} = \{\varepsilon\}^{(c)} + \{\varepsilon\}^{(r)}$

$\theta_\varepsilon \neq \theta_\varepsilon^{(c)} \neq \theta_\varepsilon^{(r)}$

$\theta_\varepsilon^{(c)} = \theta_\varepsilon^{(r)}$ if $\theta_r = \theta_\sigma$ and $\alpha_r = \pi/2$ (local variation neglected)

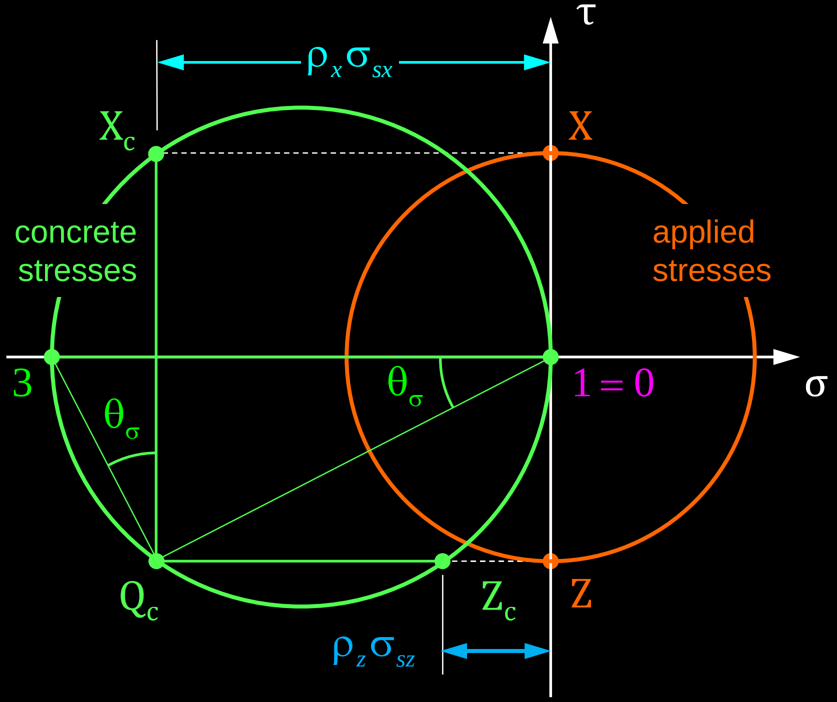
Compression field models

	Classic compression field model	Modified compression field theory (MCFT)	Cracked membrane model with rotating cracks (CMM-R)	Cracked membrane model with fixed cracks (CMM-F)
Main assumptions	Stress-free rotating cracks $\theta_\sigma = \theta_\varepsilon = \theta_r$ $\sigma_{c1} = 0$	"Stress-free" rotating cracks $\theta_\sigma = \theta_\varepsilon = \theta_r$ $\sigma_{c1m}(\varepsilon_1) > 0$ (avg. tension stiff.)	Stress-free rotating cracks $\theta_\sigma = \theta_\varepsilon = \theta_r$ $\sigma_{c1r} = 0$	Fixed interlocked cracks $\theta_{\sigma r} \neq \theta_\varepsilon \neq \theta_r$ $\sigma_{c1r} \neq 0$ (aggregate interlock)
Equilibrium	(3 equations)	in average stresses (3 equations)	at the crack (3 equations)	at the cracks (7 equations)
Compression softening	neglected (ultimate load overestimated)	considered	considered	considered
Tension stiffening	neglected (stiffness underestimated)	as average concrete property (lack of consistency)	according to tension chord model	according to tension chord model
Crack spacing	$s_r \rightarrow 0$	s_r can**not** be estimated	s_r can be estimated	s_r can be estimated
Deformation capacity	**cannot** be estimated	**cannot** be consistently estimated	**can** be estimated	**can** be estimated

Compression field models

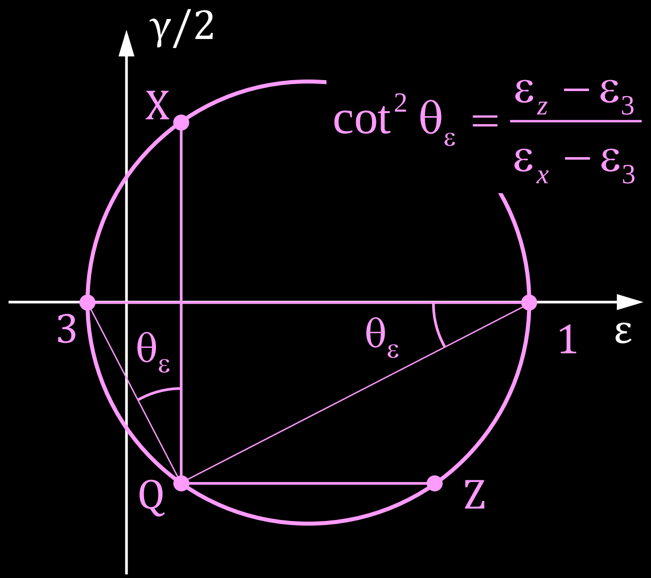
Classic compression field model with $f_{ct} = 0$ – stress-free cracks with variable crack inclination

Equilibrium



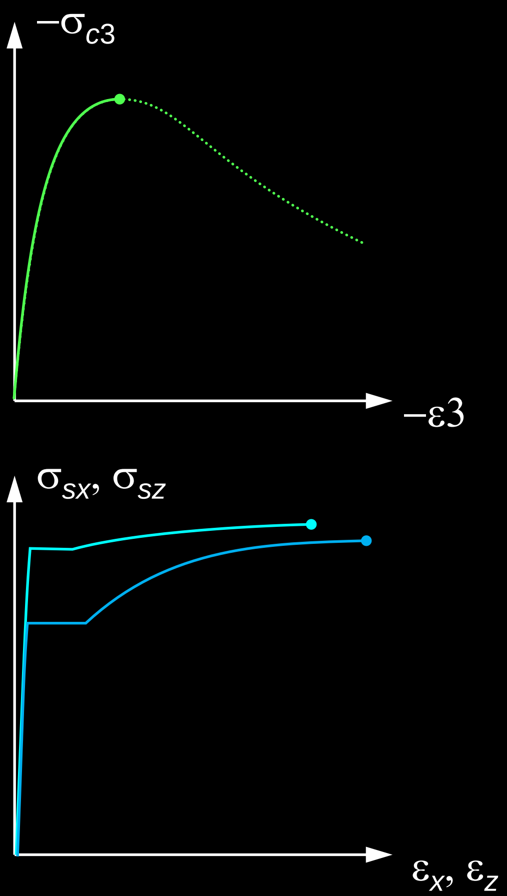
$$\sigma_x = \sigma_{c3} \cos^2 \theta_\sigma + \rho_x \sigma_{sx}$$
$$\sigma_z = \sigma_{c3} \sin^2 \theta_\sigma + \rho_z \sigma_{sz}$$
$$\tau_{xz} = -\sigma_{c3} \sin \theta_\sigma \cos \theta_\sigma$$

Compatibility



cracks parallel to θ_σ
and opening at $\alpha_r = \pi/2$
$\rightarrow \theta_\varepsilon = \theta_\sigma$

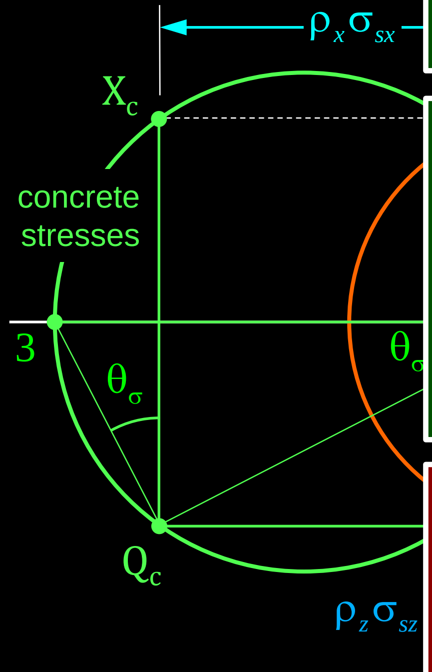
Material properties



Compression field models

Classic compress... ...ation

Equilibrium ... ...es



$$\sigma_x = \sigma_{c3} \cos^2 \theta_\sigma + \rho$$
$$\sigma_z = \sigma_{c3} \sin^2 \theta_\sigma + \rho_z$$
$$\tau_{xz} = -\sigma_{c3} \sin \theta_\sigma \cos$$

Unique solution: 3 equations for 3 unknowns

(3 non-collinear strains as primary unknowns
e.g. ε_z, ε_z und ε_3)

Cracked elastic behaviour ($n = E_s / E_c$): analytical solution for principal direction θ [Baumann 1972]:

$$\tan^2 \theta \rho_x \left(1 + n\rho_z\right) + \tan \theta \rho_x \frac{\sigma_z}{\tau_{xz}} = \cot^2 \theta \rho_z \left(1 + n\rho_x\right) + \cot \theta \rho_z \frac{\sigma_x}{\tau_{xz}}$$

$-\varepsilon 3$

Prediction of the load-deformation behaviour:

- Ultimate load overestimated (concrete compression failure)

→Compression softening!

- Stiffness is underestimated

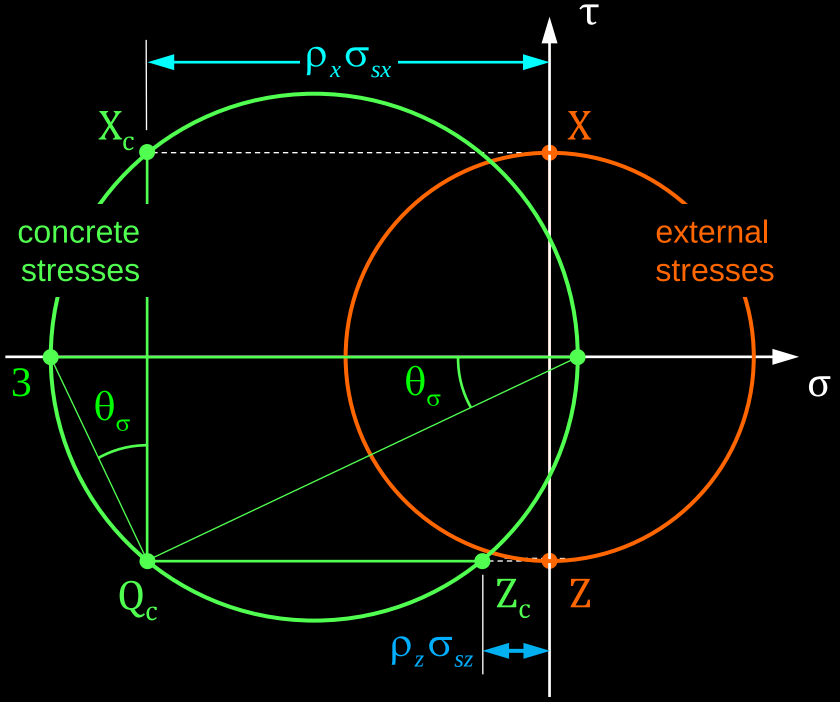
→Tension stiffening!

ε_x, ε_z

Compression field models

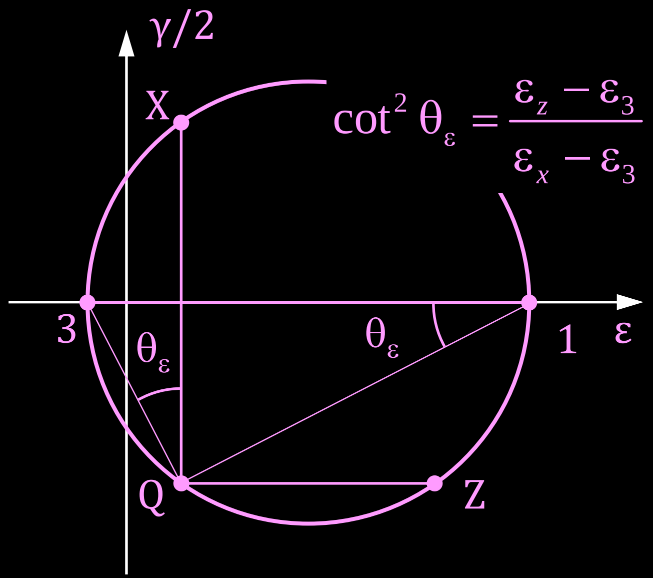
Modified compression field theory: Consideration of compression softening and tension stiffening

Equilibrium



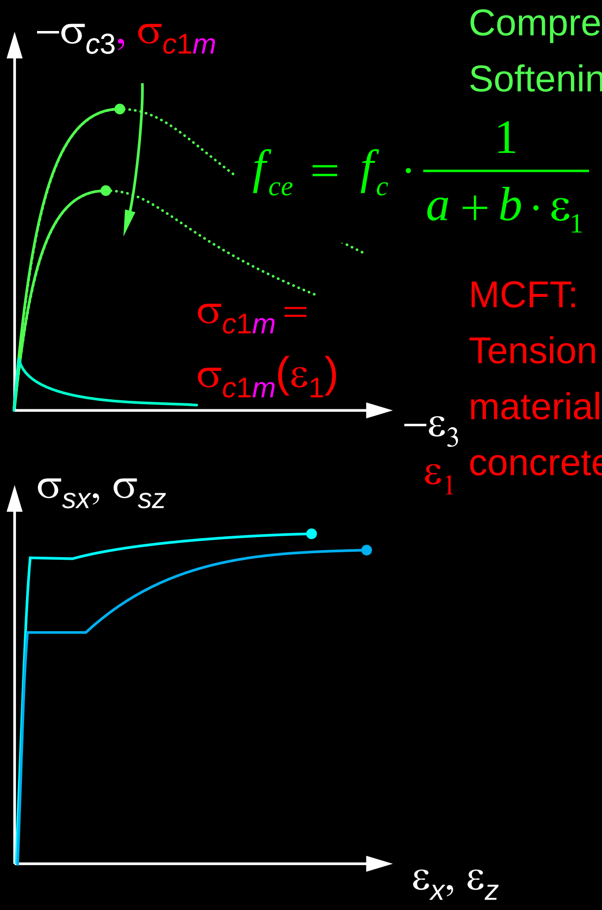
$$\sigma_x = \sigma_{c3m} \cos^2 \theta_\sigma + \sigma_{c1m} \sin^2 \theta_\sigma + \rho_x \sigma_{sxm}$$
$$\sigma_z = \sigma_{c3m} \sin^2 \theta_\sigma + \sigma_{c1m} \cos^2 \theta_\sigma + \rho_z \sigma_{szm}$$
$$\tau_{xz} = \left(\sigma_{c1m} - \sigma_{c3m}\right) \sin \theta_\sigma \cos \theta_\sigma$$

Compatibility



cracks parallel to θ_σ
and opening at $\alpha_r = \pi/2$
$\rightarrow \theta_\varepsilon = \theta_\sigma$

Material properties



Compression Softening

MCFT:
Tension stiffening as a material property of concrete

Compression field models

Modified compression field theory: Consideration of compression softening and tension stiffening

Equilibrium

Compatibility

Material properties

Consideration of the tension stiffening by “average" tensile stresses in concrete (MCFT, Vecchio & Collins, 1986) leads to good overall results, but is not fully consistent:

- Overestimation of load capacity → verification "shear at crack" (incompatible with basic assumption $\theta_\varepsilon = \theta_\sigma$)
- There is no section with equilibrium in “average" stresses
- Tension stiffening ≠ Concrete property ≠ isotropic (main influence: ρ_x, ρ_z → orthotropic)
- No information on stresses at the crack, crack spacing, etc.

conc

stre

3

$$\sigma_x = \sigma_{c3m} \cos^2\theta_\sigma + \sigma_{c1m} \sin^2\theta_\sigma + \rho_x \sigma_{sxm}$$

$$\sigma_z = \sigma_{c3m} \sin^2\theta_\sigma + \sigma_{c1m} \cos^2\theta_\sigma + \rho_z \sigma_{szm}$$

$$\tau_{xz} = (\sigma_{c1m} - \sigma_{c3m}) \sin\theta_\sigma \cos\theta_\sigma$$

cracks parallel to θ_σ

and opening at $\alpha_r = \pi/2$

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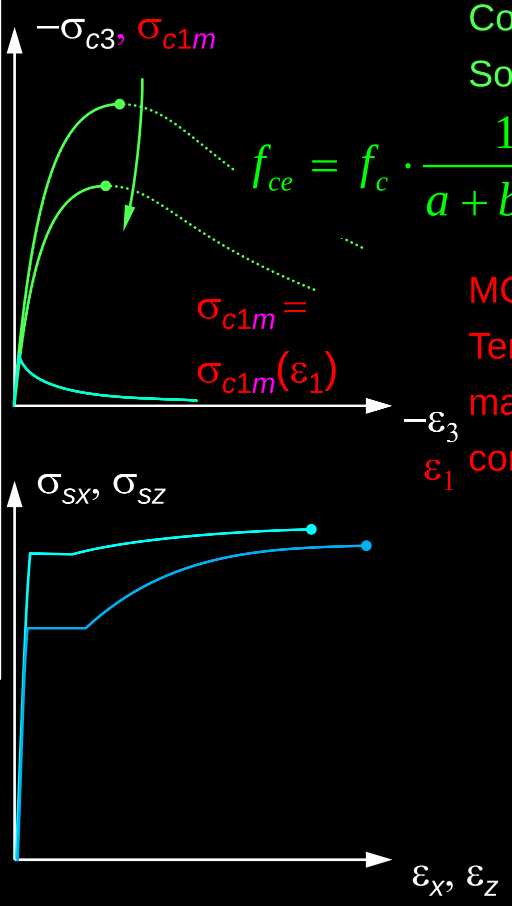


Compression Softening

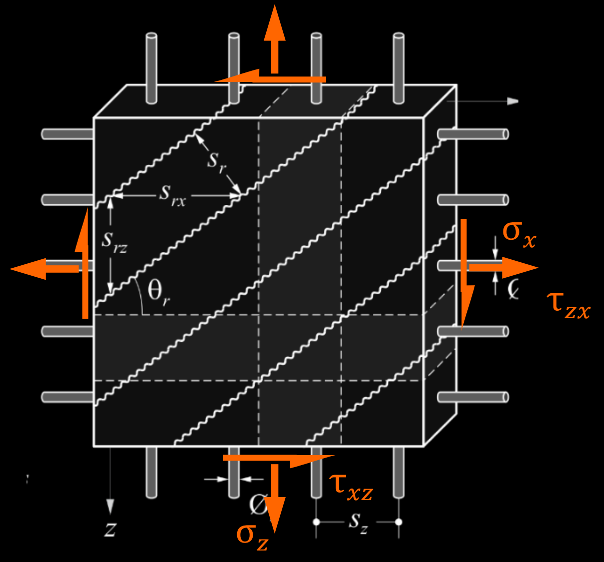
$$f_{ce} = f_c \cdot \frac{1}{a + b \cdot \varepsilon_1}$$

MCFT:

Tension stiffening as a material property of concrete

Compression field models

Cracked membrane model with rotating cracks: simplified



$$\sigma_{sxr} = \sigma_{sxr}\left(\varepsilon_x\right)$$
$$\sigma_{szr} = \sigma_{szr}\left(\varepsilon_z\right)$$

$$\sigma_x = \sigma_{c3r}\cos^2\theta_\sigma + \rho_x\sigma_{sxr}$$
$$\sigma_z = \sigma_{c3r}\sin^2\theta_\sigma + \rho_z\sigma_{szr}$$
$$\tau_{xz} = -\sigma_{c3r}\sin\theta_\sigma\cos\theta_\sigma$$

Assumption of stress-free cracks with variable crack direction

→ Stress field with uniaxial compression (parallel to crack direction) in concrete at cracks

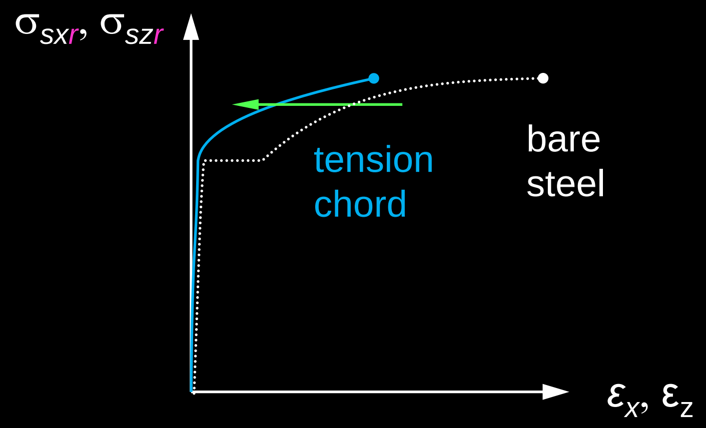
Equilibrium at the crack

→ Equations identical to the classical compression field model with $f_{ct} = 0$

Treatment of reinforcement as tension chords

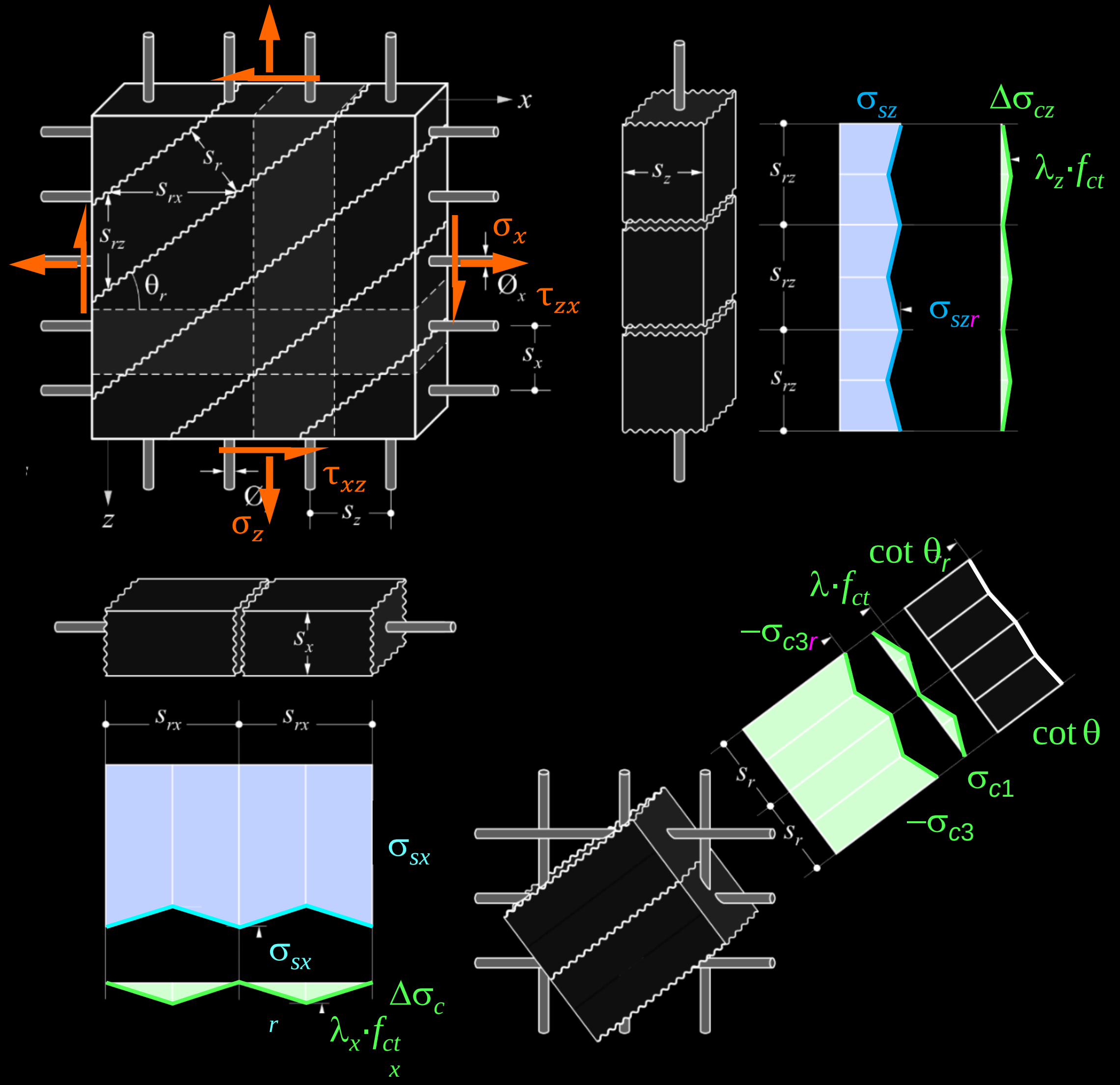
→ Tension stiffening increases stiffness, not ultimate load

→ Stress-strain relationships for stresses at crack σ_{sxr}, σ_{szr} with respect to mean strains ε_x, ε_z



Compression field models

Cracked membrane model with rotating cracks: simplified



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Treatment of reinforcement as tension chords

→ Tension stiffening increases stiffness, not ultimate load

→ Stress-strain relationships for stresses at crack σ_{sxr}, σ_{szr} as function of mean strains ε_x, ε_z

Determination of stresses in concrete and crack spacing

→ Stress in the concrete = superposition of the compression field and the stresses transferred to the concrete by bond

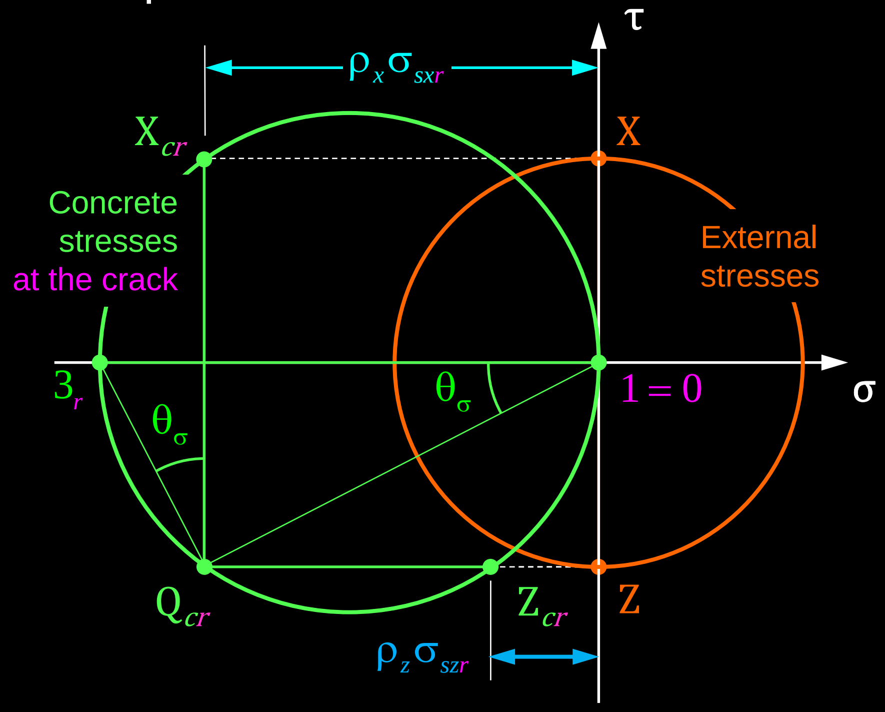
→ Condition for diagonal crack spacing: Principal tensile stress between two cracks must not exceed f_{ct}.

→ Crack spacings in the direction of reinforcement are geometrically linked to diagonal crack spacing: $s_{rx} = s_r / \sin\theta_r$, $s_{rz} = s_r / \cos\theta_r$

Compression field models

Cracked membrane model with rotating cracks: Consideration of tension stiffening and compression softening

Equilibrium

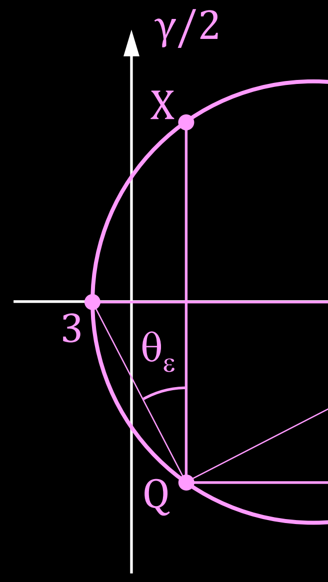


$$\sigma_x = \sigma_{c3r} \cos^2 \theta_\sigma + \rho_x \sigma_{sxr}$$

$$\sigma_z = \sigma_{c3r} \sin^2 \theta_\sigma + \rho_z \sigma_{szr}$$

$$\tau_{xz} = -\sigma_{c3r} \sin \theta_\sigma \cos \theta_\sigma$$

Compatibility



$$\cot^2 \theta_\varepsilon = \frac{\varepsilon_z - \varepsilon_3}{\varepsilon_x - \varepsilon_3}$$

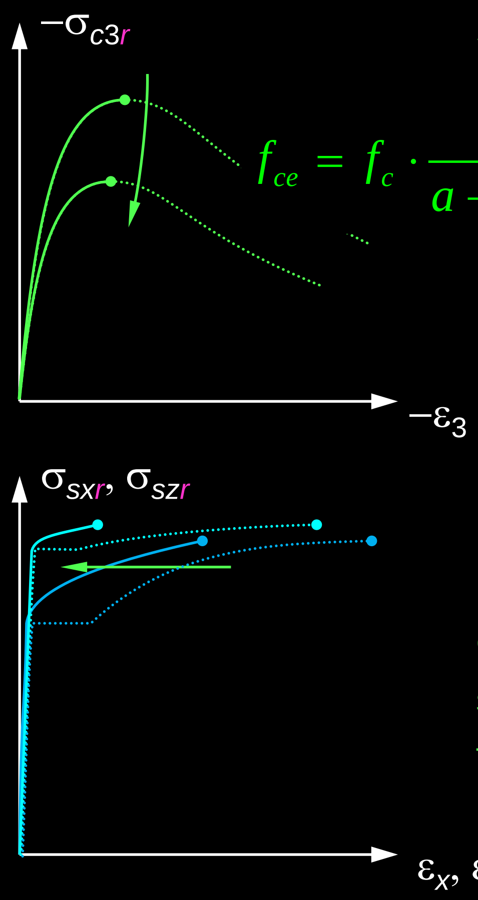
cracks parallel to θ_σ

and opening at $\alpha_r = \pi/2$

$\rightarrow \theta_\varepsilon = \theta_\sigma$

Material properties

Compression softening



$$f_{ce} = f_c \cdot \frac{1}{a + b \cdot \varepsilon_1}$$

CMM: Tension stiffening according to the tension chord model

Compression field models

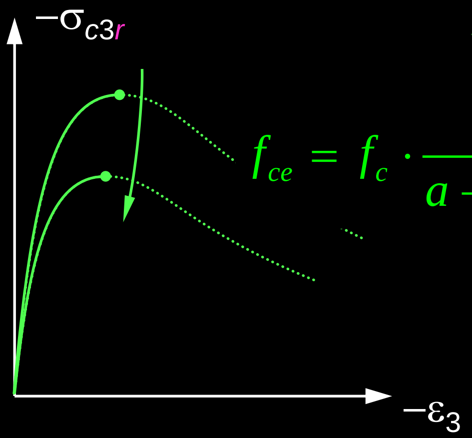
Cracked membrane model with rotating cracks: Consideration of tension stiffening and compression softening

Equilibrium

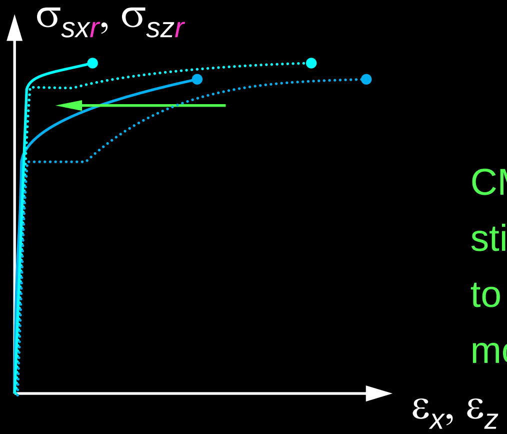
Compatibility

Material properties

Compression softening



$$f_{ce} = f_c \cdot \frac{1}{a + b \cdot \varepsilon_1}$$



CMM: Tension stiffening according to the tension chord model

Cond stres at the c

3_r

Consideration of tension stiffening via modified stress-strain relationship of the reinforcement (CMM, Kaufmann & Marti 1998):

$\rightarrow$ Equilibrium formulated in stresses at crack "r", consistent with basic assumption

$\rightarrow$ Direct information on maximum stresses at the crack, crack spacing etc.

$\rightarrow$ Direct link to limit analysis

$\rightarrow$ Good prediction of load-deformation behaviour

σ_x

$\sigma_z = \sigma_{c3r} \sin^2 \theta_\sigma + \rho_z \sigma_{szr}$

$\tau_{xz} = -\sigma_{c3r} \sin \theta_\sigma \cos \theta_\sigma$

and opening at $\alpha_r = \pi/2$

$\rightarrow \theta_\varepsilon = \theta_\sigma$

Compression field models

Cracked membrane model with rotating cracks: Consideration of tension stiffening and compression softening

Equilibrium

τ

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σ

$$\sigma_z = \sigma_{c3r} \sin^2 \theta_\sigma + \rho_z \sigma_{szr}$$

$$\tau_{xz} = -\sigma_{c3r} \sin\theta_\sigma \cos\theta_\sigma$$

Compatibility

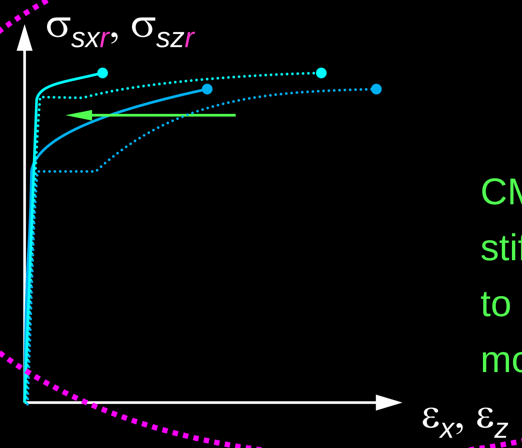
and opening at $\alpha_r = \pi/2$

$\rightarrow \theta_\varepsilon = \theta_\sigma$

Material properties

Stress-strain relationship required for stresses at crack as a function of average strains

→ Tension chord model



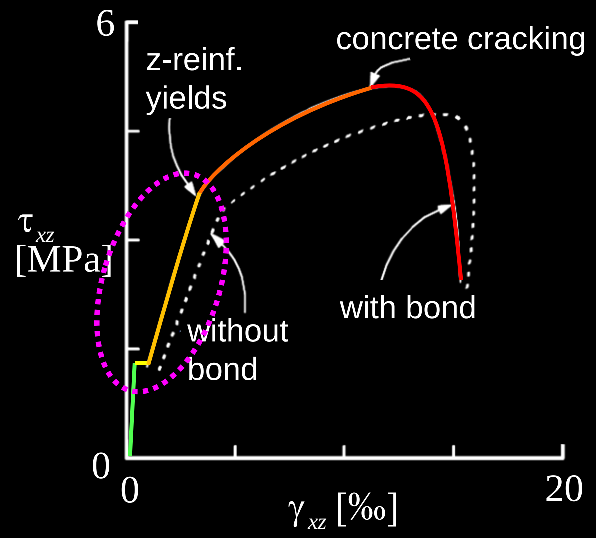
CMM: Tension stiffening according to the tension chord model

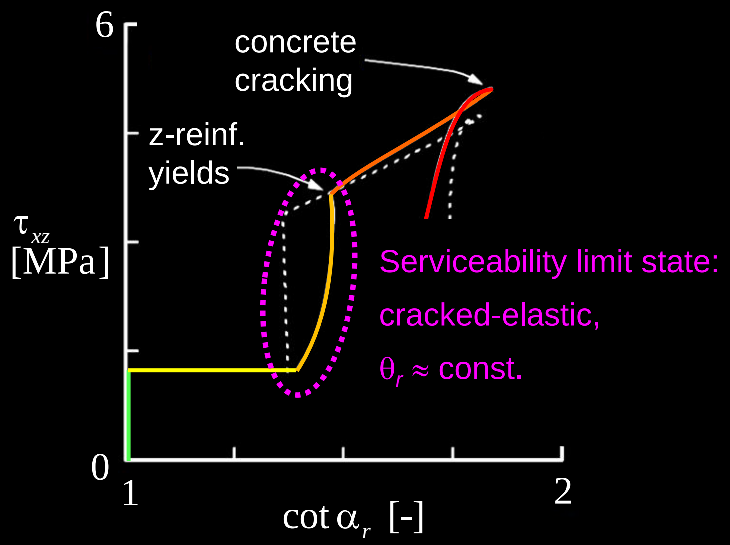
Membrane elements - Load-deformation behaviour

Reinforced concrete membrane element under monotonous load increase

1. Uncracked behaviour: Like homogeneous concrete membrane element (slight differences due to restraint shrinkage etc.)
2. Initial cracking approximately perpendicular to the principal tensile stress direction
3. Crack formation → Redistribution of internal forces → Change of principal stress directions immediately after crack formation
4. Cracked-elastic behaviour: Principal stress directions ± constant as long as both reinforcements remain elastic
5. ~~Yielding of a reinforcemen~~
   ~~→ Decrease in stiffness –~~ ~~nent of internal forces~~
   ~~→ New cracks (closer to t~~ ~~on-yielding reinforcement)~~
6. ~~Failure due to failure of the~~ ~~g of the other reinforcement (possibly re~~ ~~s or aggregate interlock fails)~~

irrelevant for serviceability limit state

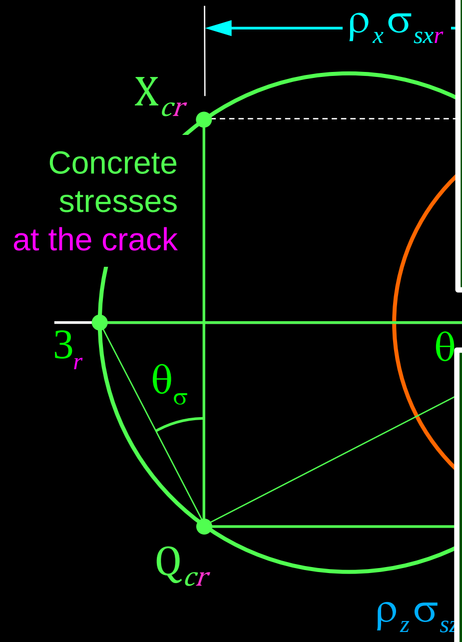




Compression field models

Cracked membrane model with rotating cracks

Equilibrium | Compatibility | Material properties



Unique solution: 3 equations for 3 unknowns

(3 non-collinear strains as primary unknowns e.g. ε_x, ε_z, and ε_3)

Cracked elastic behaviour ($n = E_s / E_c$): analytical solution for θ_r (with $f_{ct} = 0 \rightarrow$ same as Baumann 1972):

$$\tan^2\theta_r\rho_x(1+n\rho_z)+\tan\theta_r\rho_x\left\{\frac{\sigma_z}{\tau_{xz}}-\frac{f_{ct}}{2\tau_{xz}}\left[\lambda_z+n\rho_z\left(\lambda_x+\frac{n-1}{n}\lambda_z-\lambda\right)\right]\right\}=$$

$$=\cot^2\theta_r\rho_z(1+n\rho_x)+\tan\theta_r\rho_z\left\{\frac{\sigma_x}{\tau_{xz}}-\frac{f_{ct}}{2\tau_{xz}}\left[\lambda_x+n\rho_x\left(\lambda_z+\frac{n-1}{n}\lambda_x-\lambda\right)\right]\right\}$$

$$\sigma_x = \sigma_{c3r}\cos^2\theta_\sigma + \ldots$$

$$\sigma_z = \sigma_{c3r}\sin^2\theta_\sigma + \ldots$$

$$\tau_{xz} = -\sigma_{c3r}\sin\theta_\sigma\cos\theta_\sigma$$

$\rightarrow \theta_\varepsilon = \theta_\sigma$

Compression Softening

$$f_c \cdot \frac{1}{a + b \cdot \varepsilon_1}$$

$-\varepsilon_3$

CMM: Tension stiffening according to the tension chord model

ε_x, ε_z

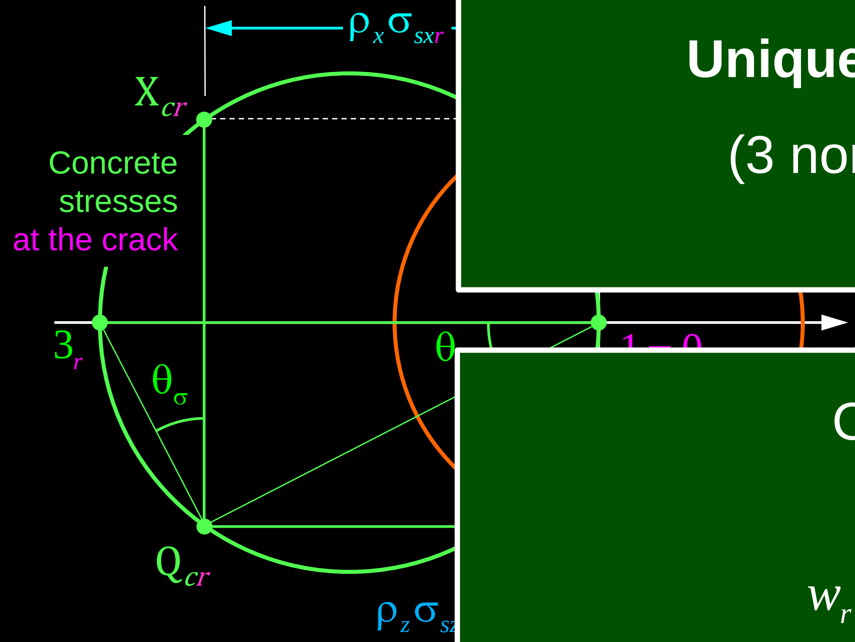
Compression field models

Cracked membrane model with rotating cracks

Equilibrium

Compatibility

Material properties



Unique solution: 3 equations for 3 unknowns

(3 non-collinear strains as primary unknowns e.g. ε_x, ε_z and ε_3)

Compression Softening

$$f_c \cdot \frac{1}{a + b \cdot \varepsilon_1}$$

Crack widths result from strains and diagonal crack spacing s_r :

$$w_r = s_r\left(\varepsilon_1 - \varepsilon_{c1}\right) = \lambda s_{r0}\left(\varepsilon_1 - \frac{\lambda f_{ctm}}{2E_c}\right) \approx \lambda s_{r0}\varepsilon_1$$

crack spacing

total strain (incl. tension stiffening)

deformation of the concrete between the cracks

$$\sigma_x = \sigma_{c3r}\cos^2\theta_\sigma + \rho_x \sigma_{sxr}$$

$$\sigma_z = \sigma_{c3r}\sin^2\theta_\sigma + \rho_z \sigma_{szr}$$

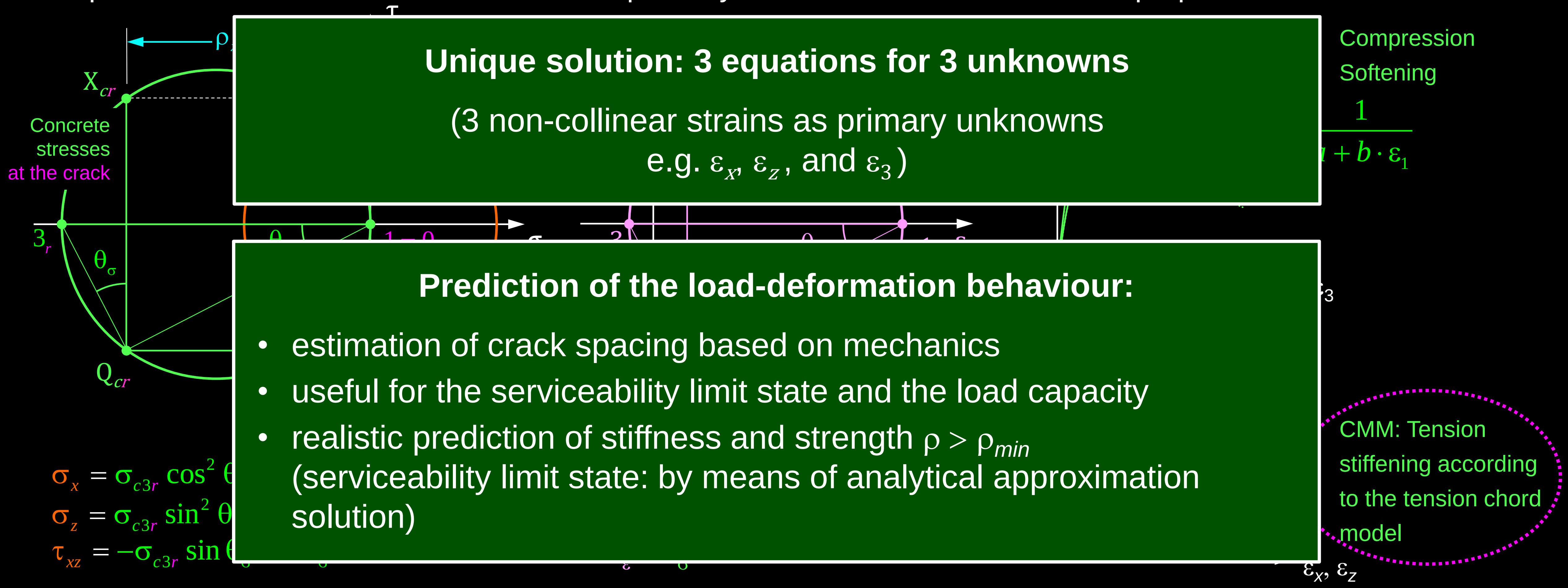
$$\tau_{xz} = -\sigma_{c3r}\sin\theta_\sigma\cos\theta_\sigma$$

CMM: Tension stiffening according to the tension chord model

Compression field models

Cracked membrane model with rotating cracks

Equilibrium | Compatibility | Material properties



Unique solution: 3 equations for 3 unknowns

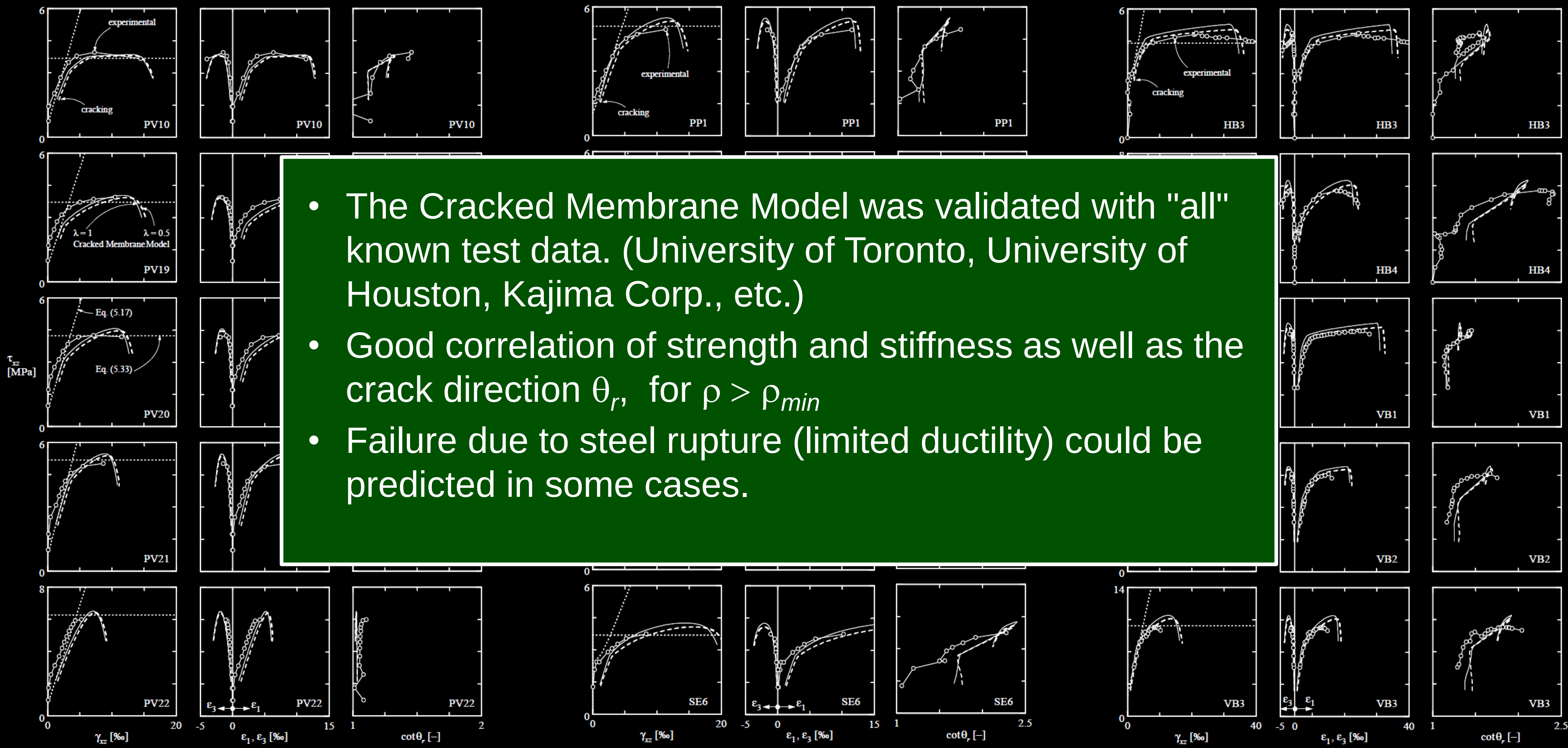
(3 non-collinear strains as primary unknowns e.g. ε_x, ε_z, and ε_3)

Prediction of the load-deformation behaviour:

- estimation of crack spacing based on mechanics
- useful for the serviceability limit state and the load capacity
- realistic prediction of stiffness and strength $\rho > \rho_{min}$ (serviceability limit state: by means of analytical approximation solution)

Compression field models

Cracked membrane model with rotating cracks: Comparison with experiment: load-deformation behaviour



- The Cracked Membrane Model was validated with "all" known test data. (University of Toronto, University of Houston, Kajima Corp., etc.)
- Good correlation of strength and stiffness as well as the crack direction θ_r, for $\rho > \rho_{min}$
- Failure due to steel rupture (limited ductility) could be predicted in some cases.

Compression field models

Cracked membrane model with rotating cracks: Application limits / open questions

Fictitious, rotating, stress-free cracks vs real, interlocking cracks

- Unsatisfactory prediction for $\rho < \rho_{min}$, no convergence for uniaxial reinforcement

$\rightarrow$ General cracked membrane model considers fixed, interlocking cracks

$\rightarrow$ Most general solution for:

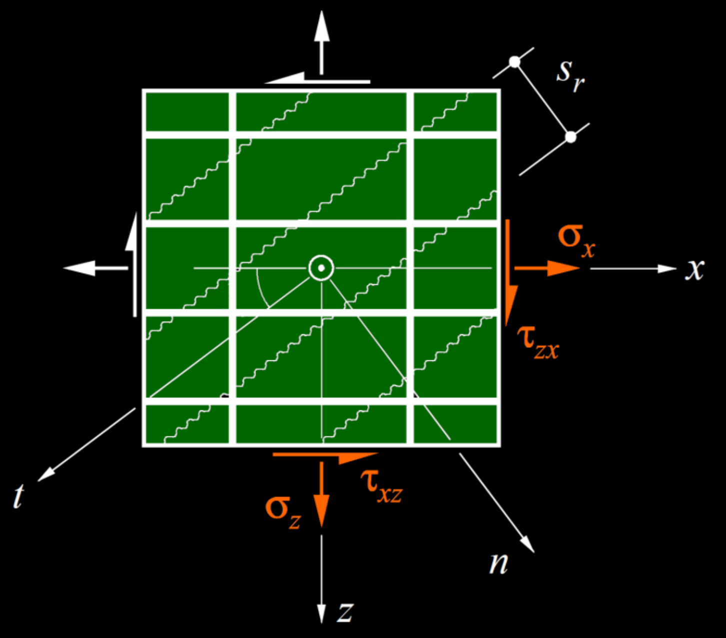
  – Only one group of parallel cracks with equal distances over the entire element

  – Reinforcement is considered as equivalent stress (constant over rebar spacing and membrane element thickness).

Compression field models

Cracked membrane model with fixed cracks: General solution, with aggregate interlock

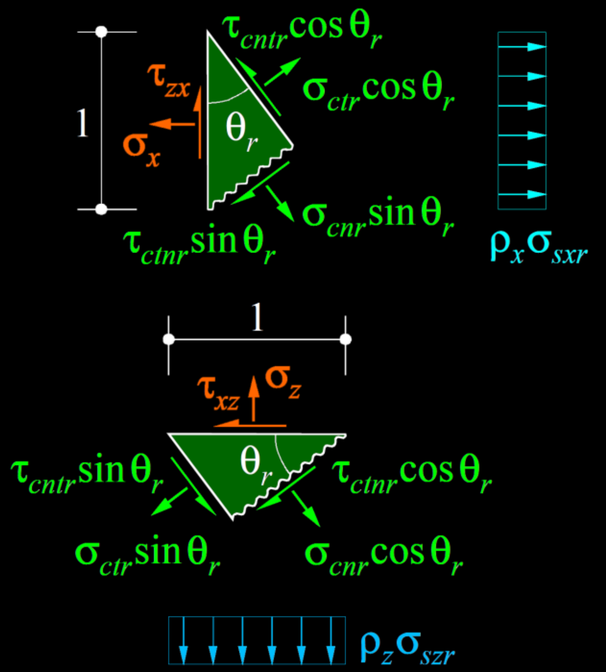
Membrane element



Required material properties:

- Constitutive relationships of concrete and reinforcement
- Bond-slip relationship

Stresses at crack / equilibrium

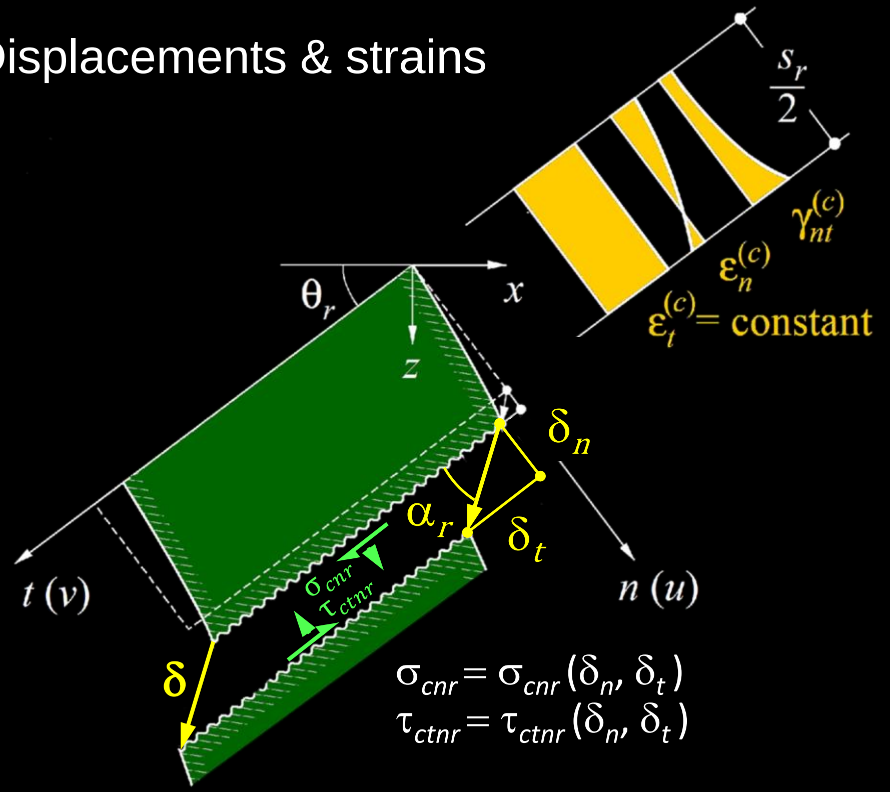


$$\sigma_x = \rho_x\sigma_{sxr} + \sigma_{cnr}\sin^2\theta_r + \sigma_{ctr}\cos^2\theta_r - \tau_{ctnr}\sin(2\theta_r)$$

$$\sigma_z = \rho_z\sigma_{szr} + \sigma_{cnr}\cos^2\theta_r + \sigma_{ctr}\sin^2\theta_r + \tau_{ctnr}\sin(2\theta_r)$$

$$\tau_{xz} = (\sigma_{cnr} - \sigma_{ctr})\sin\theta_r\cos\theta_r - \tau_{ctnr}\cos(2\theta_r)$$

Displacements & strains



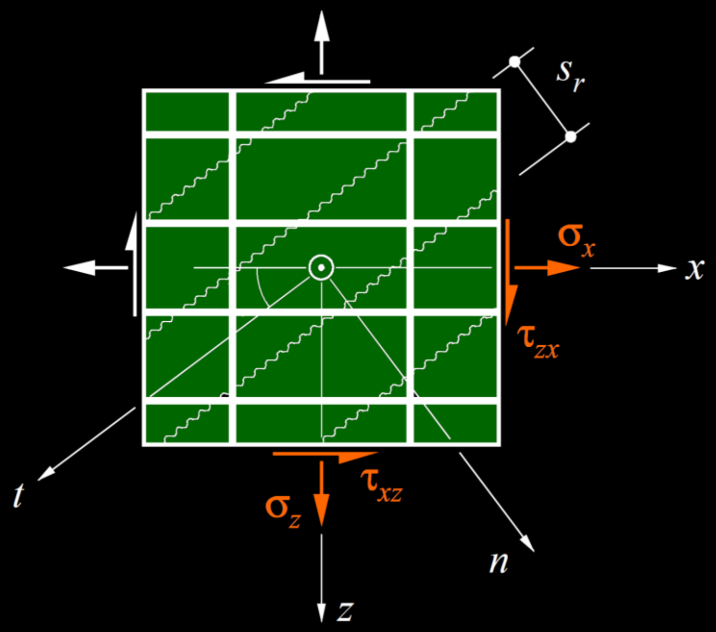
$\varepsilon_n^{(c)}$, $\varepsilon_t^{(c)}$, $\gamma_{nt}^{(c)}$ are independent of the coordinate t; thus $\partial\gamma_{nt}^{(c)}/\partial t = 0$, i.e. $\partial\varepsilon_t^{(c)}/\partial n = 0$ and $\varepsilon_t^{(c)} =$ constant

($\varepsilon_n = \partial u/\partial n$, $\varepsilon_t = \partial v/\partial t$, $\gamma_{nt} = \partial u/\partial t + \partial v/\partial n$)

Compression field models

Cracked membrane model with fixed cracks: General solution, with aggregate interlock

Membrane element



Required material properties:

- Constitutive relationships of concrete and reinforcement
- Bond-slip relationship

General solution method (for given crack inclination and spacing)

Assumption / estimation of 7 primary unknowns:

- 3 stress components at crack σ_{sxr}, σ_{szr}, σ_{ctr}
- 2 crack displacements (opening and slip) δ_n, δ_t
- 2 concrete displacements at the crack u_{cr}, v_{cr}

Determine the concrete stresses at the crack σ_{cnr}, τ_{ctnr} via the crack opening and slip δ_n, δ_t using the aggregate interlock relationship $\sigma_{cnr} = \sigma_{cnr}(\delta_n, \delta_t)$, $\tau_{ctnr} = \tau_{ctnr}(\delta_n, \delta_t)$.

The bond stress as well as the stresses, strains, and displacements in the concrete and reinforcement are determined by means of the differential equilibrium and the compatibility conditions. This is done starting from the crack ($n = s_r/2$), in infinitesimal steps dn going towards $n = 0$.

Iteration until the following conditions are met (7 equations for 7 unknowns):

- 3 equilibrium conditions at the crack
- 2 components of the concrete displacements u_c, v_c and 2 reinforcement displacements u_{sx}, u_{sz} must vanish in the middle between two cracks.

Compression field models

Cracked membrane model with fixed cracks: Application limits / open questions

Lack of experimental data (measured directly, not biased by the measurement)

- Stresses in concrete cannot be measured experimentally (they are usually estimated by surface strains).
- Local measurements of the stresses in the steel with conventional instrumentation (e.g. with strain gauges, ...) depend on the location of the measurement (near or far from the crack). In addition, they usually disturb the bond.

→ The most commonly used relationships for tension stiffening and compression softening have been insufficiently validated with experiments.

→ Today, it is possible to measure the steel strains continuously along an embedded reinforcing bar using fibre optic strain sensing without disturbing bond; new insights from experimental testing of panels

- Crack kinematics (in particular the crack slip) are difficult to record with conventional instrumentation (unless the location of the cracks is known in advance); only limited experimental data are available.
- Push-off tests are not necessarily representative of aggregate interlock in biaxially reinforced elements.

→ Aggregate interlock relationship still needs to be validated.

→ Today, with 3D Digital Image Correlation (DIC) and Automatic Crack Detection & Measurement of their kinematics (ACDM) new insights into the behaviour are gained

Compatibility and deformation capacity of membrane elements: Summary

Summary

- The Cracked Membrane Model (general formulation with aggregate interlock) requires (numerical) solving of seven highly nonlinear equations with seven unknown quantities: very complex
- Simplification with Cracked Membrane Model (without aggregate interlock) = combination of the classic compression field models with the tension chord model:
  - Stress-free cracks parallel to the direction of the principal strains (variable crack direction, fictitious cracks)
  - Tension stiffening effect of the concrete between the cracks according to the tension chord model (without influence on resistance of reinforcement, indirect influence on ultimate load as strains become smaller → higher concrete compressive strength)
  - Concrete compressive strength as a function of strain state (transverse strain)

- The Cracked Membrane Model (without aggregate interlock) generally provides good agreement with test results. In the serviceability limit state (elastic reinforcement), the analytical approximation solution can be easily applied.
- The consideration of the aggregate interlock (general formulation of the Cracked Membrane Model) would make sense if the element is only reinforced in one direction or if the reinforcement ratio is very low in the other direction.

Annex

Shear Panel Tester, University of Toronto (1979)

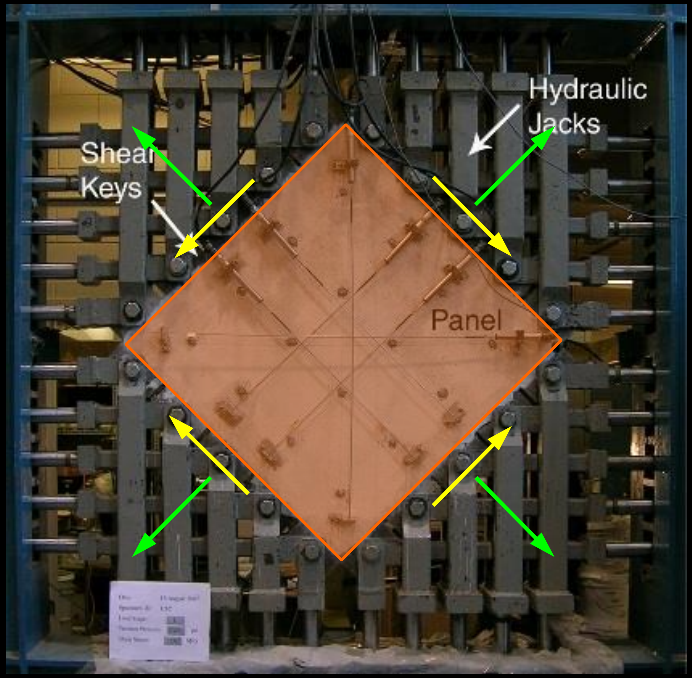
In-plane loading (3 stress resultants)

Applied in-plane loads
perpendicular and parallel to element edge

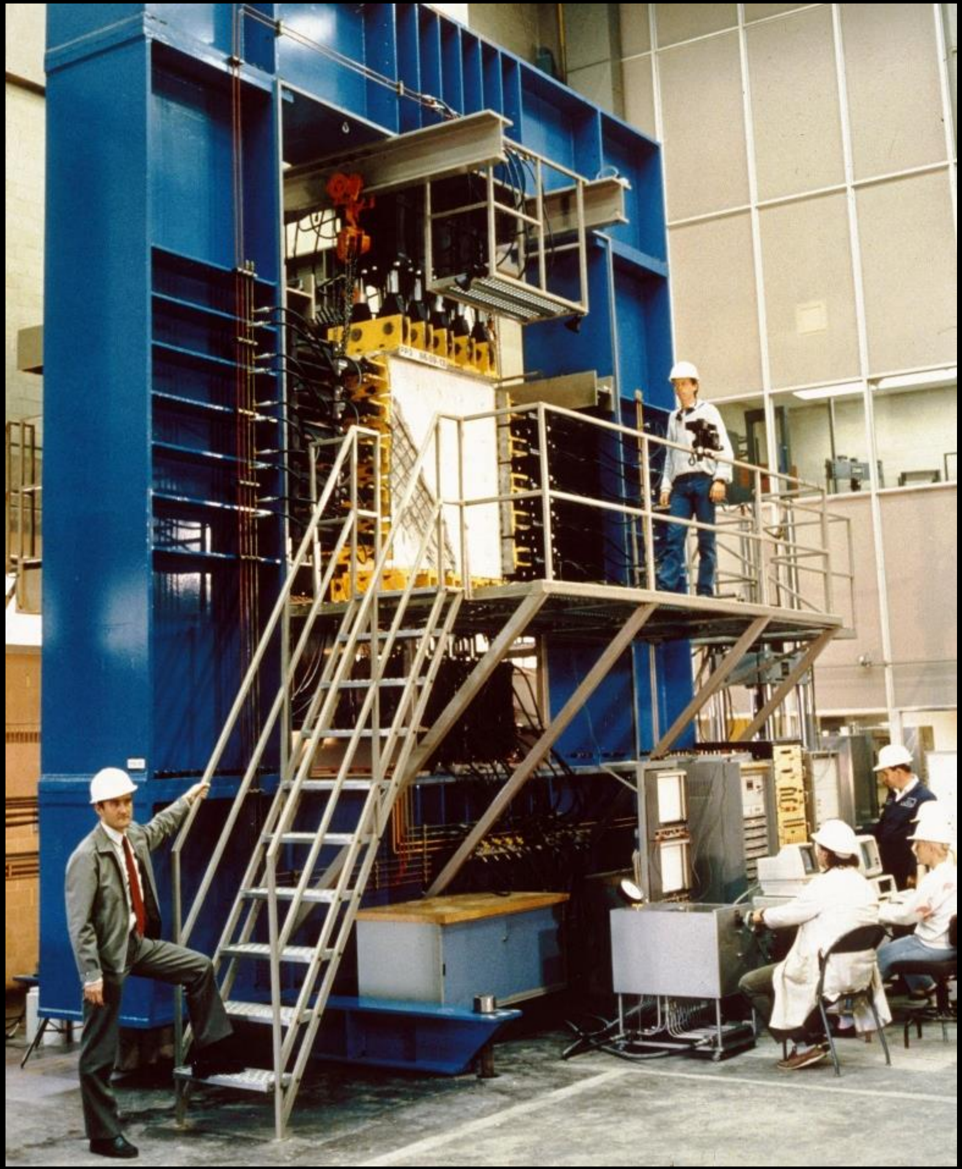
→ principal direction of applied loads variable

→ reinforcing bars parallel to element edges

Element size 890-890-70 mm



Shell Element Tester, University of Toronto (1984 / 2009)



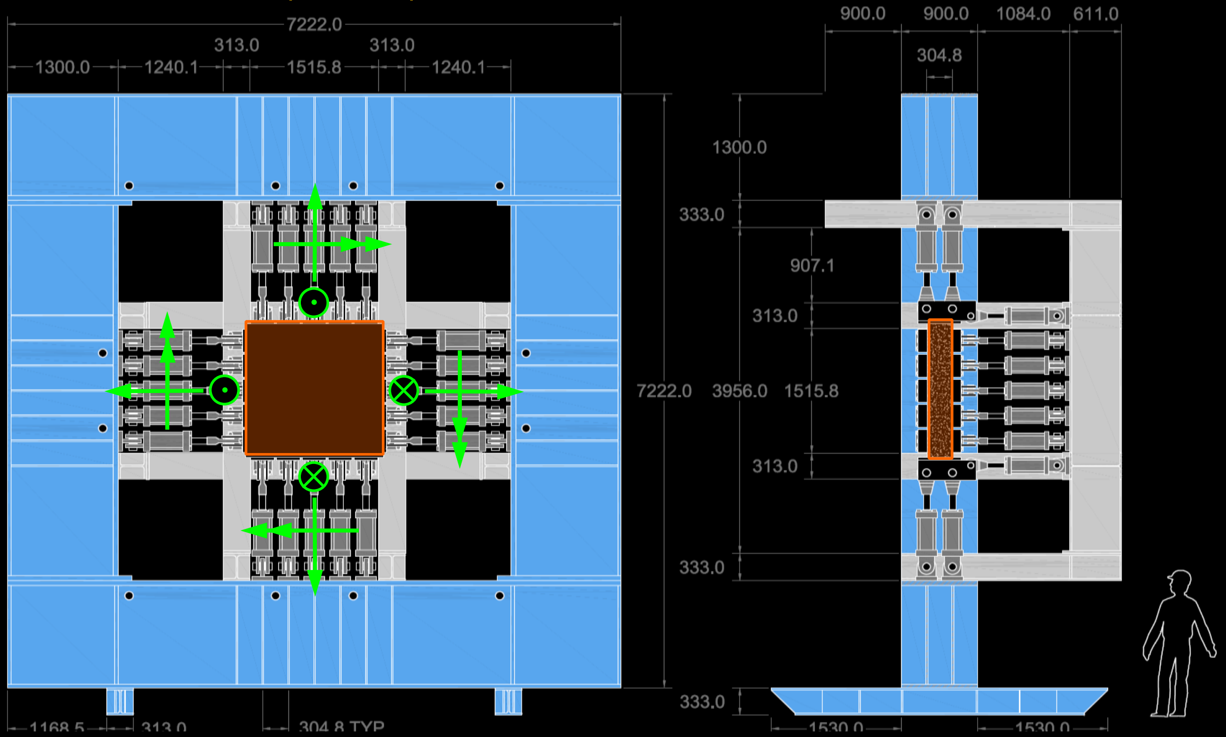
General loading (8 stress resultants)

Applied loads in-plane and out-of-plane, perpendicular to element edge

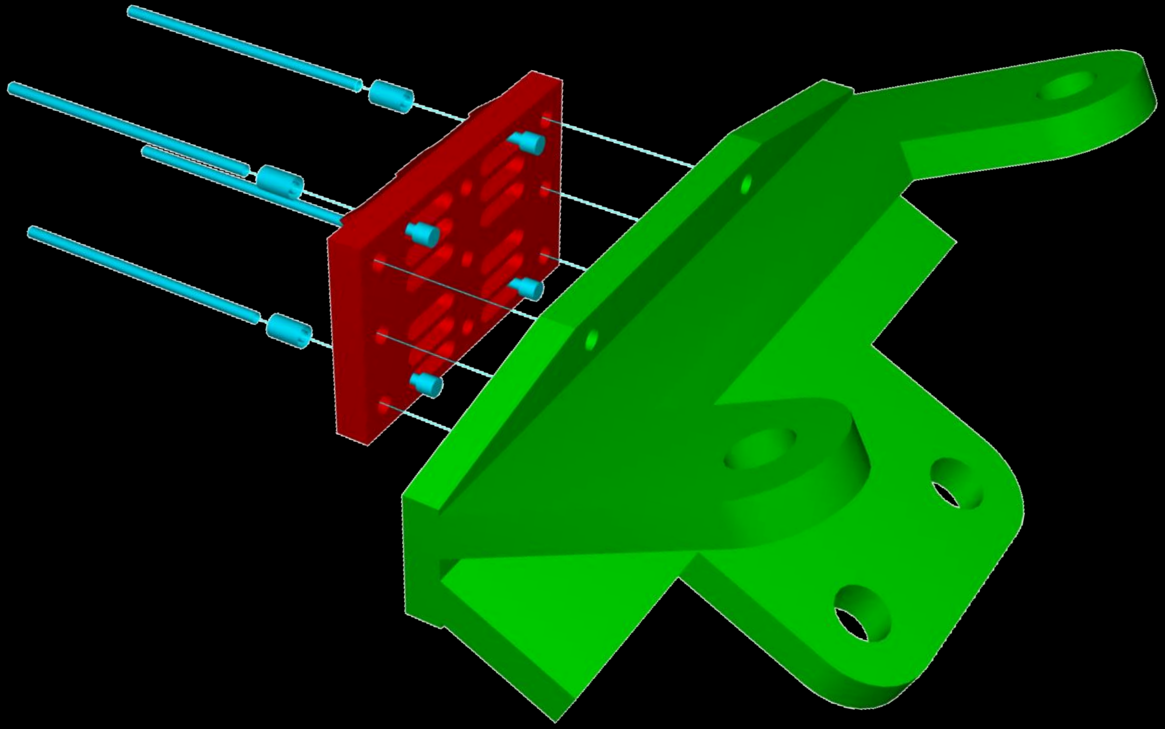
→ principal direction of applied loads constant

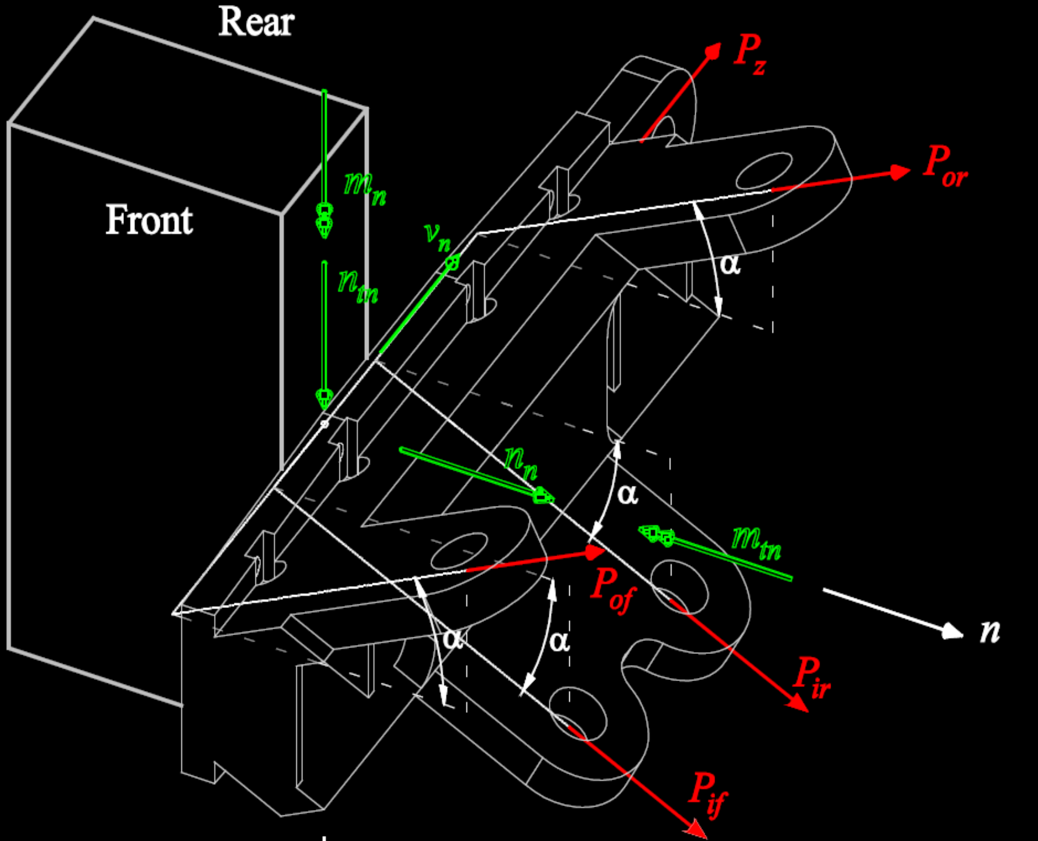
→ reinforcing bars at angle to element edges

Element size 1,524-1,524-350 mm



Large Universal Shell Element Tester LUSET, ETH Zurich (2017)





Load introduction

20 yokes, 20 blocks bolted to yokes

reinforcing bars with threaded ends and bar couplers (e.g. Bartec)

$$v_n = \frac{P_z}{0.4\,\mathrm{m}}$$

$$n_n = \frac{P_{of} + P_{if} + P_{or} + P_{ir}}{0.4\,\mathrm{m}} \cdot \cos\alpha$$

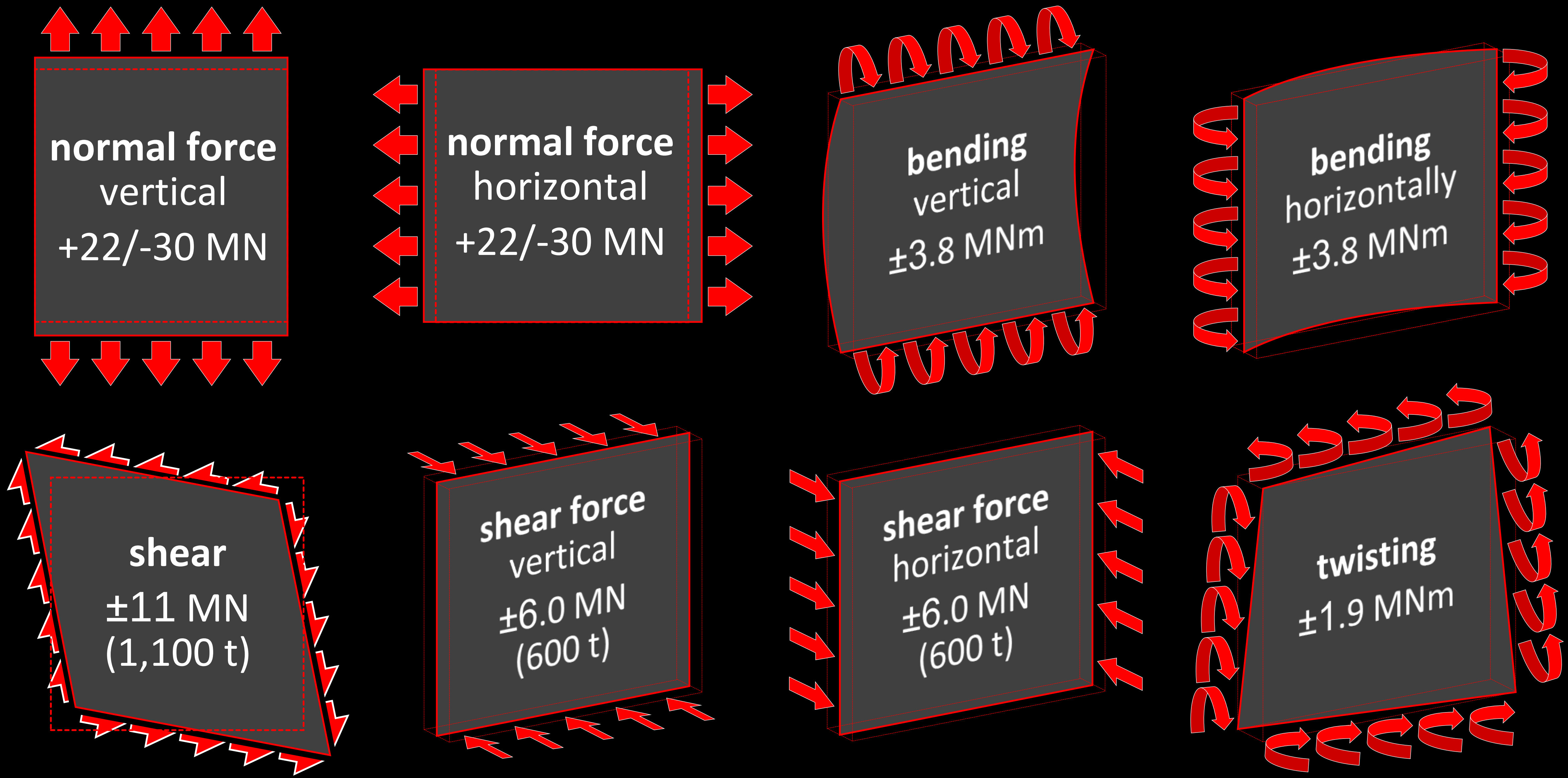
$$n_{tn} = \frac{-P_{of} + P_{if} - P_{or} + P_{ir}}{0.4\,\mathrm{m}} \cdot \sin\alpha$$

t

$$m_n = \frac{\left(P_{or} - P_{of}\right)\cdot 1.5e + \left(P_{ir} - P_{if}\right)\cdot 0.5e}{0.4\,\mathrm{m}} \cdot \cos\alpha$$

$$m_{tn} = \frac{-\left(P_{or} - P_{of}\right)\cdot 1.5e + \left(P_{ir} - P_{if}\right)\cdot 0.5e}{0.4\,\mathrm{m}} \cdot \sin\alpha$$

Large Universal Shell Element Tester LUSET, ETH Zurich (2017)



Large Universal Shell Element Tester LUSET, ETH Zurich (2017)

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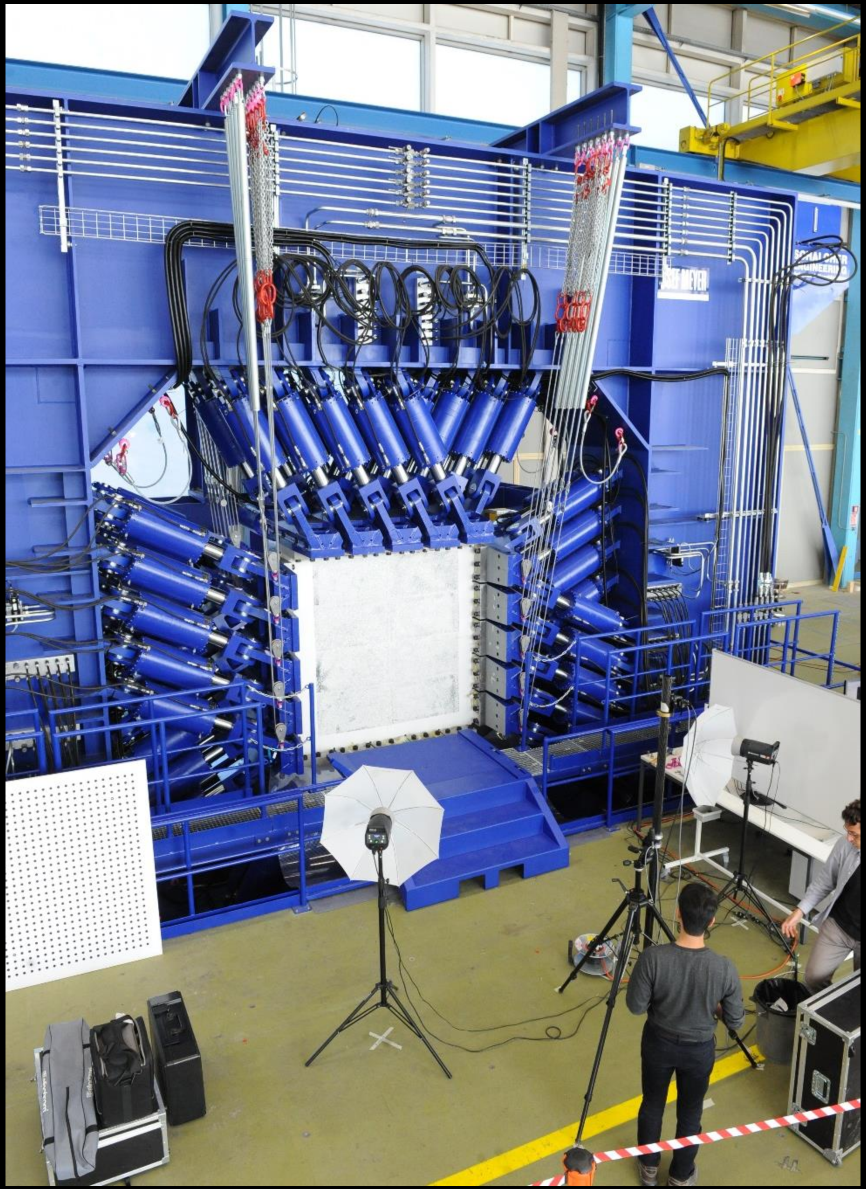
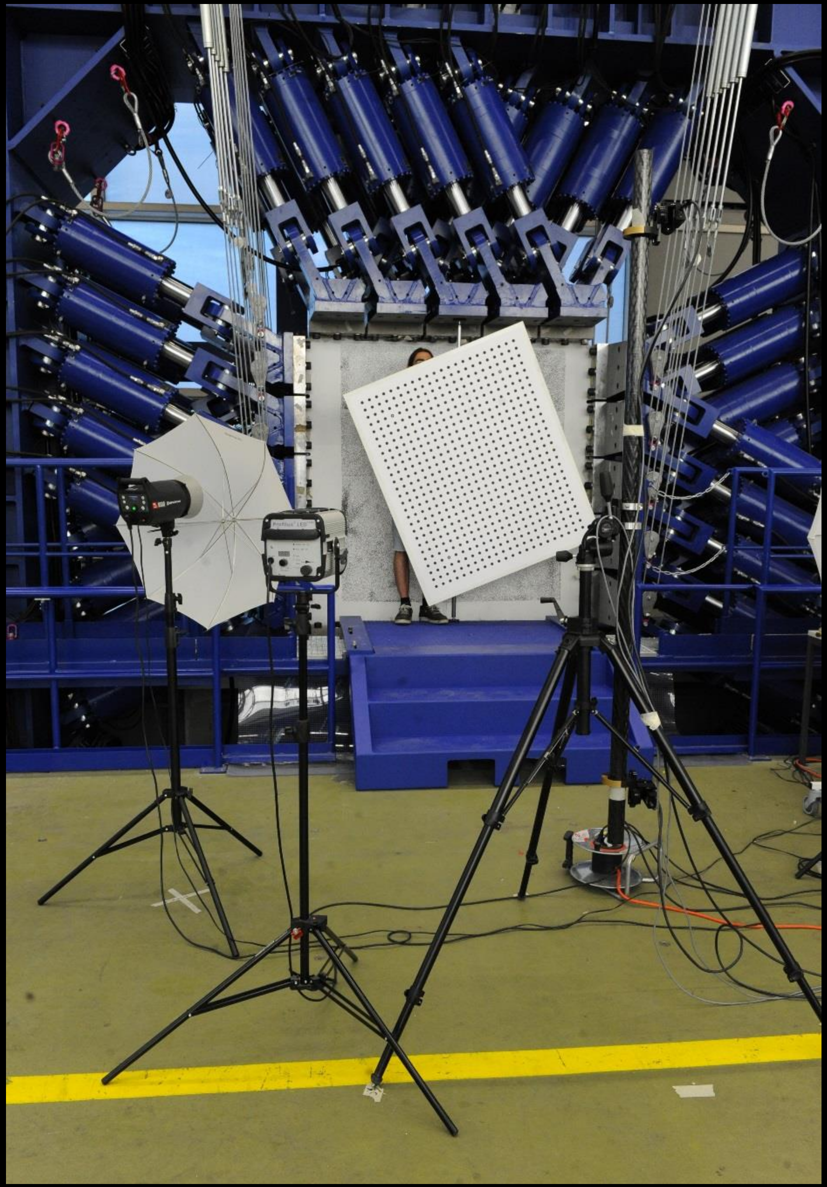
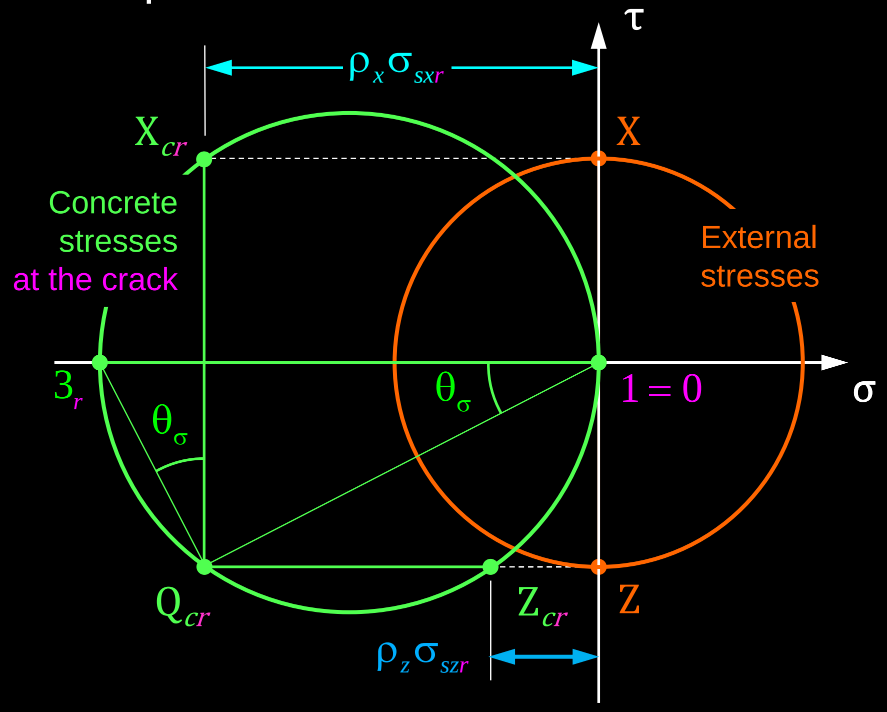




Compression field models

Cracked membrane model with rotating cracks: Consideration of tension stiffening and compression softening

Equilibrium

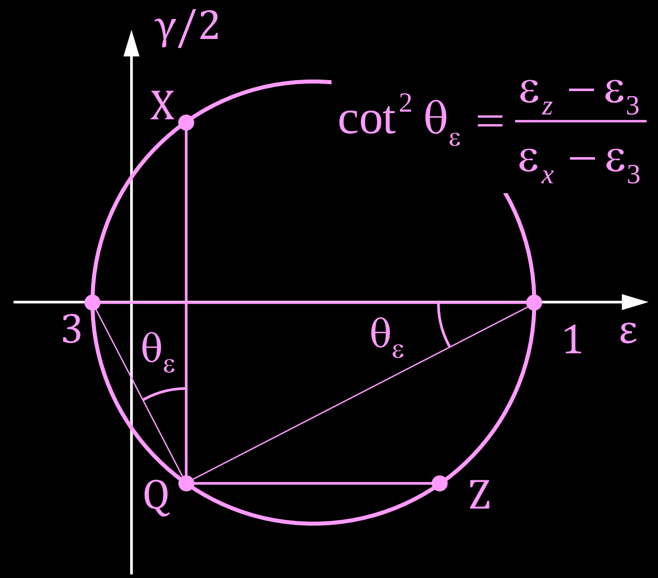


$$\sigma_x = \sigma_{c3r} \cos^2 \theta_\sigma + \rho_x \sigma_{sxr}$$

$$\sigma_z = \sigma_{c3r} \sin^2 \theta_\sigma + \rho_z \sigma_{szr}$$

$$\tau_{xz} = -\sigma_{c3r} \sin \theta_\sigma \cos \theta_\sigma$$

Compatibility



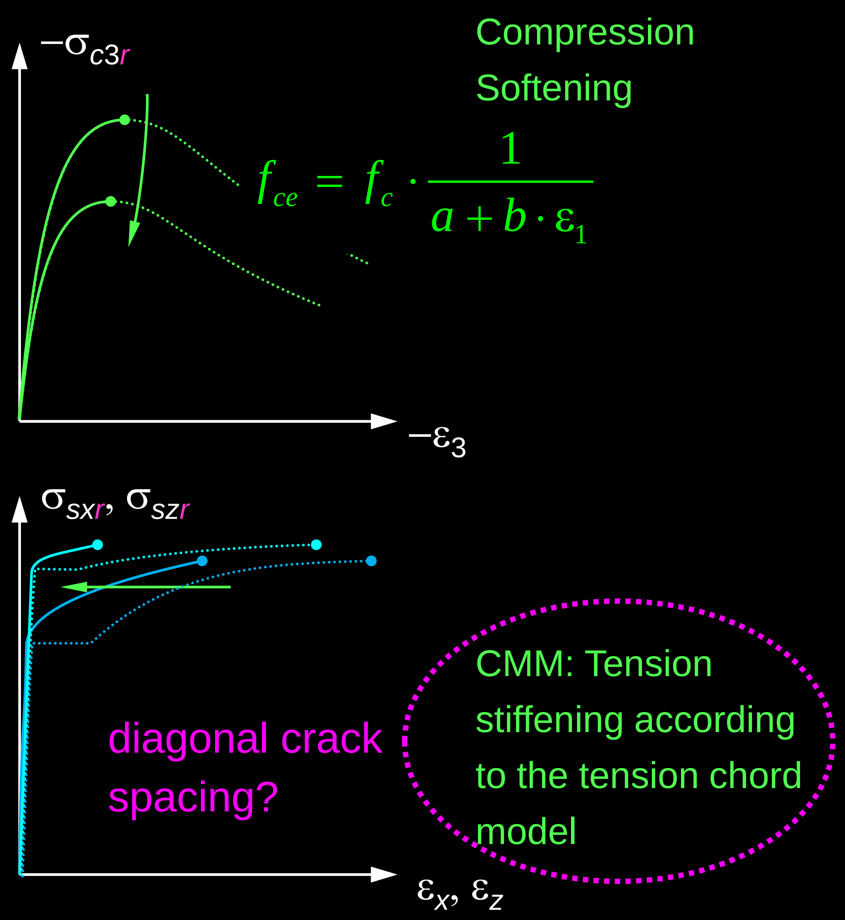
$$\cot^2 \theta_\varepsilon = \frac{\varepsilon_z - \varepsilon_3}{\varepsilon_x - \varepsilon_3}$$

cracks parallel to θ_σ

and opening at $\alpha_r = \pi/2$

$\rightarrow \theta_\varepsilon = \theta_\sigma$

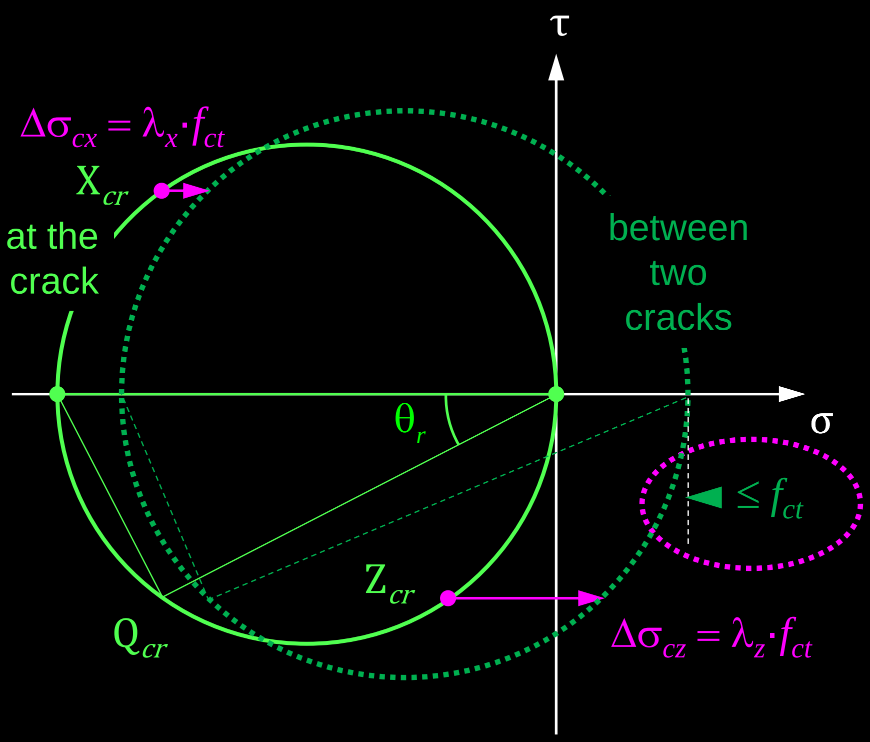
Material properties



Compression field models

Cracked membrane model with rotating cracks: Determination of the maximum diagonal crack spacing

Exact solution



Maximum crack spacing for uniaxial tension in reinforcement direction: s_{rx0}, s_{rz0} (according to the tension chord model)

$$s_{rx0} = \frac{f_{ct}\varnothing_x}{2\tau_{b0}}\frac{(1-\rho_x)}{\rho_x}$$

$$s_{rz0} = \frac{f_{ct}\varnothing_z}{2\tau_{b0}}\frac{(1-\rho_z)}{\rho_z}$$

Geometric relationship between s_{rx}, s_{rz} and diagonal crack spacing s_r

$$s_r = s_{rx}\sin\theta_r = s_{rz}\cos\theta_r$$

Parameters for crack distance $\lambda = 0.5...1$:

($\lambda = 1.0$: max. crack distance $s_r = s_{r0}$

$\lambda = 0.5$: min. crack distance $s_r = s_{r0}/2$)

$$\lambda = s_r / s_{r0}$$

$$\lambda_x = \frac{\Delta\sigma_{cx}}{f_{ct}} = \frac{s_{rx}}{s_{rx0}} = \frac{s_r}{s_{rx0}\sin\theta_r}$$

$$\lambda_z = \frac{\Delta\sigma_{cz}}{f_{ct}} = \frac{s_{rz}}{s_{rz0}} = \frac{s_r}{s_{rz0}\cos\theta_r}$$

Principal stress σ_{c1} between two cracks:

$$\sigma_{c1} = \frac{f_{ct}}{2}(\lambda_x+\lambda_z) - \frac{\tau_{xz}}{2}(\cot\theta_r+\tan\theta_r) + \sqrt{\left[\frac{\tau_{xz}}{2}(\cot\theta_r-\tan\theta_r) - \frac{f_{ct}}{2}(\lambda_x-\lambda_z)\right]^2 + \tau_{xz}^2} \leq f_{ct}$$

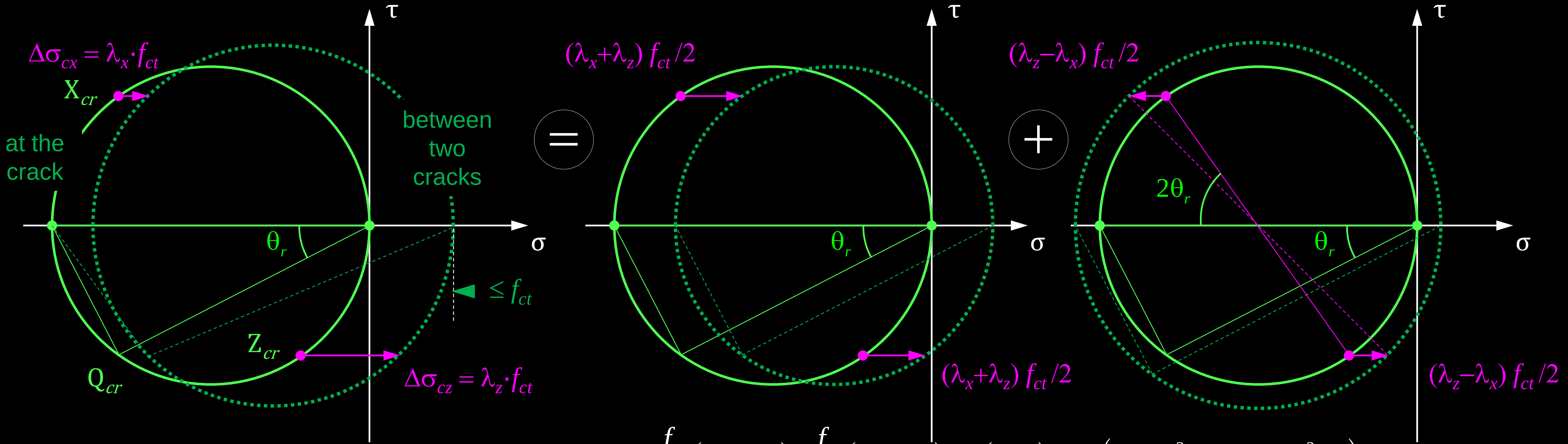
→ quadratic equation for maximum diagonal crack spacing s_{r0}

Compression field models

Cracked membrane model with rotating cracks: Determination of the maximum diagonal crack spacing

Exact solution

Approximation (symmetric / antisymmetric part of the composite)



$$\sigma_{c1} \approx \frac{f_{ct}}{2}(\lambda_x+\lambda_z) - \frac{f_{ct}}{2}(\lambda_x-\lambda_z)\cos(2\alpha_r) = f_{ct}\left(\lambda_x \sin^2\alpha_r + \lambda_z \cos^2\alpha_r\right) \leq f_{ct}$$

Closed form approximate solution for maximum diagonal crack distance s_{r0}:

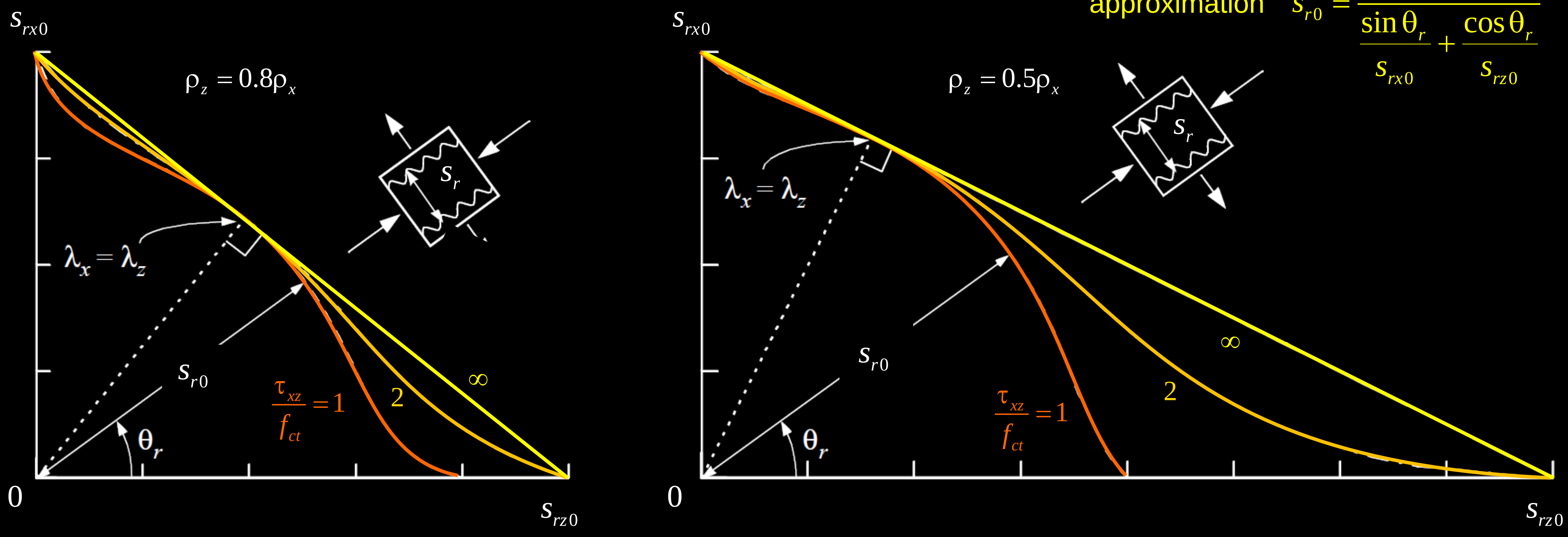
$$s_{r0} \approx \frac{1}{\dfrac{\sin\theta_r}{s_{rx0}} + \dfrac{\cos\theta_r}{s_{rz0}}}$$

Compression field models

Cracked membrane model with rotating cracks: Determination of the maximum diagonal crack spacing

exact solution (function of τ_{xz}/f_{ct})

approximation $s_{r0} = \dfrac{1}{\dfrac{\sin\theta_r}{s_{rx0}} + \dfrac{\cos\theta_r}{s_{rz0}}}$



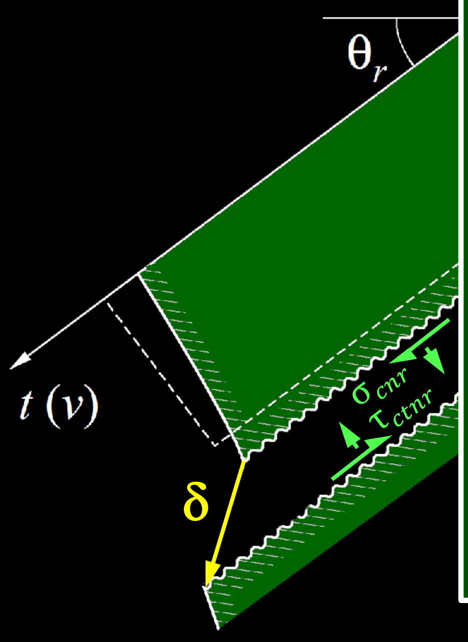
Compression field models

Cracked membrane model with rotating cracks: strains in cracked membrane elements

Total strains $\{\varepsilon\}$ = strains in concrete between cracks $\{\varepsilon\}^{(c)}$ + strains due to crack kinematics $\{\varepsilon\}^{(r)}$

Contribution to total strain:

- $\{\varepsilon\}^{(c)}$ (average c
- $\{\varepsilon\}^{(r)}$ (smeared c



$\gamma/2$

1

ε

Crack widths result from strains and diagonal crack spacing s_r :

$$\{\varepsilon\} = \{\varepsilon\}^{(c)} + \{\varepsilon\}^{(r)} \rightarrow \{\varepsilon\}^{(r)} = \{\varepsilon\} - \{\varepsilon\}^{(c)}$$

$$\rightarrow w_r = s_r \cdot \varepsilon_1^{(r)} = s_r\left(\varepsilon_1 - \varepsilon_{c1}\right)$$

$$\rightarrow w_r = \lambda s_{r0}\left(\varepsilon_1 - \frac{\lambda f_{ctm}}{2E_c}\right) \approx \lambda s_{r0}\varepsilon_1$$

(valid for $\theta_\varepsilon^{(c)} = \theta_\varepsilon^{(r)}$ and $\alpha_r = \dfrac{\pi}{2}$)

$\theta_\varepsilon = \theta_\sigma$

Total strains $\{\varepsilon\}$:

$\{\varepsilon\} = \{\varepsilon\}^{(c)} + \{\varepsilon\}^{(r)}$

$\theta_\varepsilon \neq \theta_\varepsilon^{(c)} \neq \theta_\varepsilon^{(r)}$

$\theta_\varepsilon^{(c)} = \theta_\varepsilon^{(r)}$ if $\theta_r = \theta_\sigma$ and $\alpha_r = \pi/2$ (local variation neglected)

Compression field models

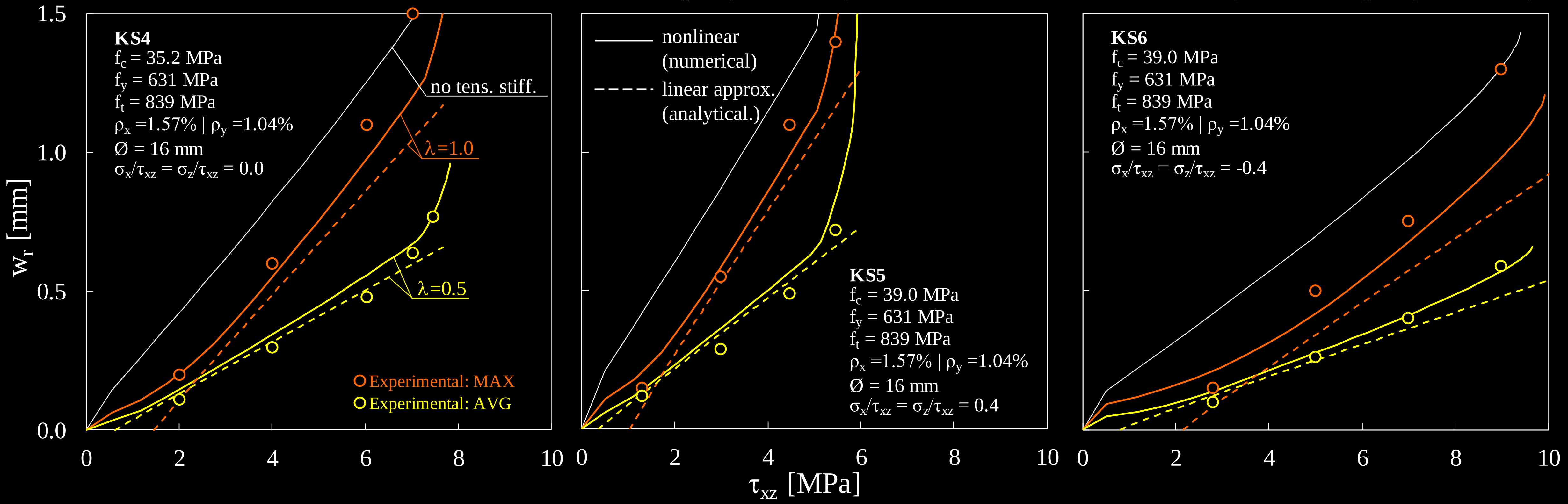
Cracked membrane model with rotating cracks: Comparison with experiments: crack widths

Tests by Proestos (2014): membrane elements 1525-1525-355 mm under uniform load

KS4: pure shear

KS5: shear and biaxial tension (proportional)

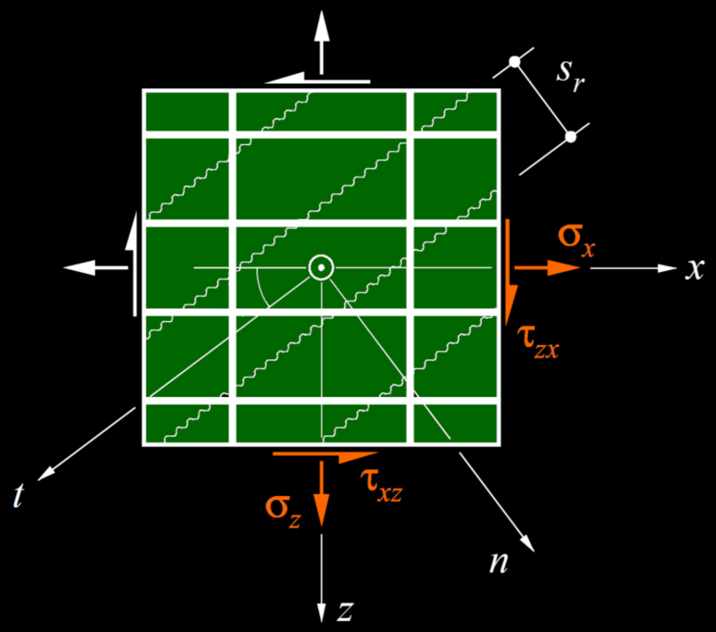
KS6: shear and biaxial compression (proportional)



Compression field models

Cracked membrane model with fixed cracks: General solution, with aggregate interlock

Membrane element



Required material properties:

- Constitutive relationships of concrete and reinforcement
- Bond-slip relationship

Simplified solution method (for given crack inclination and spacing)

Approximate the local variation of the concrete strains $\varepsilon_n^{(c)}$, $\varepsilon_t^{(c)}$, $\gamma_{nt}^{(c)}$ between the cracks based on the TCM

Assumption / estimation of 5 primary unknowns:

- Strains in concrete between two cracks $\{\varepsilon\}^{(c)} = 3$ unknowns)
- Strains due to crack kinematics $\{\varepsilon\}^{(r)} = 2$ unknowns (for known crack direction and distances, $\{\varepsilon\}^{(r)}$ follows from crack opening and crack slip δ_n, δ_t)

Iteration until the following conditions are met (5 equations for 5 unknowns):

- 3 equilibrium conditions at the crack
- 2 aggregate interlock relationships $\sigma_{cnr}(\{\varepsilon\}^{(c)}) = \sigma_{cnr}(\delta_n, \delta_t)$, $\tau_{ctnr}(\{\varepsilon\}^{(c)}) = \tau_{ctnr}(\delta_n, \delta_t)$

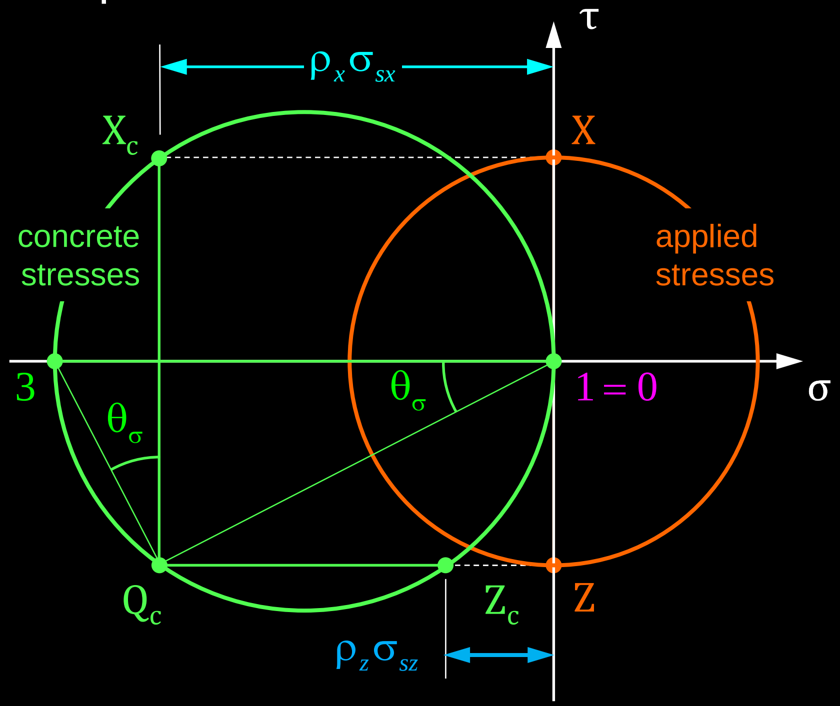
Despite the simplification of neglecting the variable concrete strains, the solution is numerically challenging, since the crack interrelationship is highly non-linear and sensitive to small displacements.

It was recently implemented successfully (Gehri 2018) and gives good results.

Measured or calculated?

Determination of stress and strain state in experiments

Equilibrium

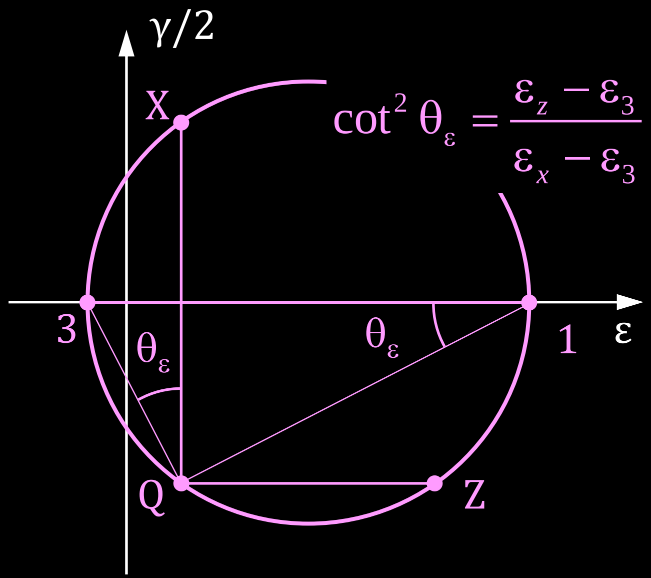


$$\sigma_x = \sigma_{c3}\cos^2\theta_\sigma + \rho_x\sigma_{sx}$$

$$\sigma_z = \sigma_{c3}\sin^2\theta_\sigma + \rho_z\sigma_{sz}$$

$$\tau_{xz} = -\sigma_{c3}\sin\theta_\sigma\cos\theta_\sigma$$

Compatibility

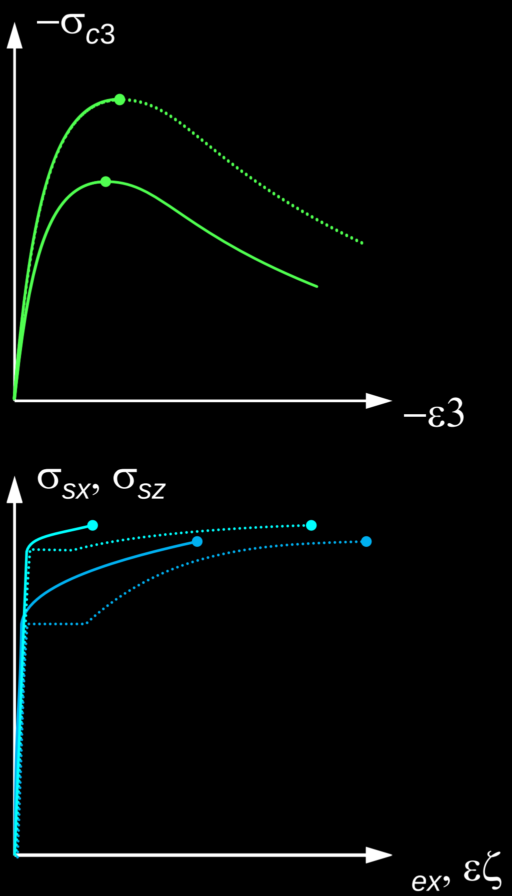


cracks parallel to θ_σ

and opening at $\alpha_r = \pi/2$

$\rightarrow \theta_\varepsilon = \theta_\sigma$

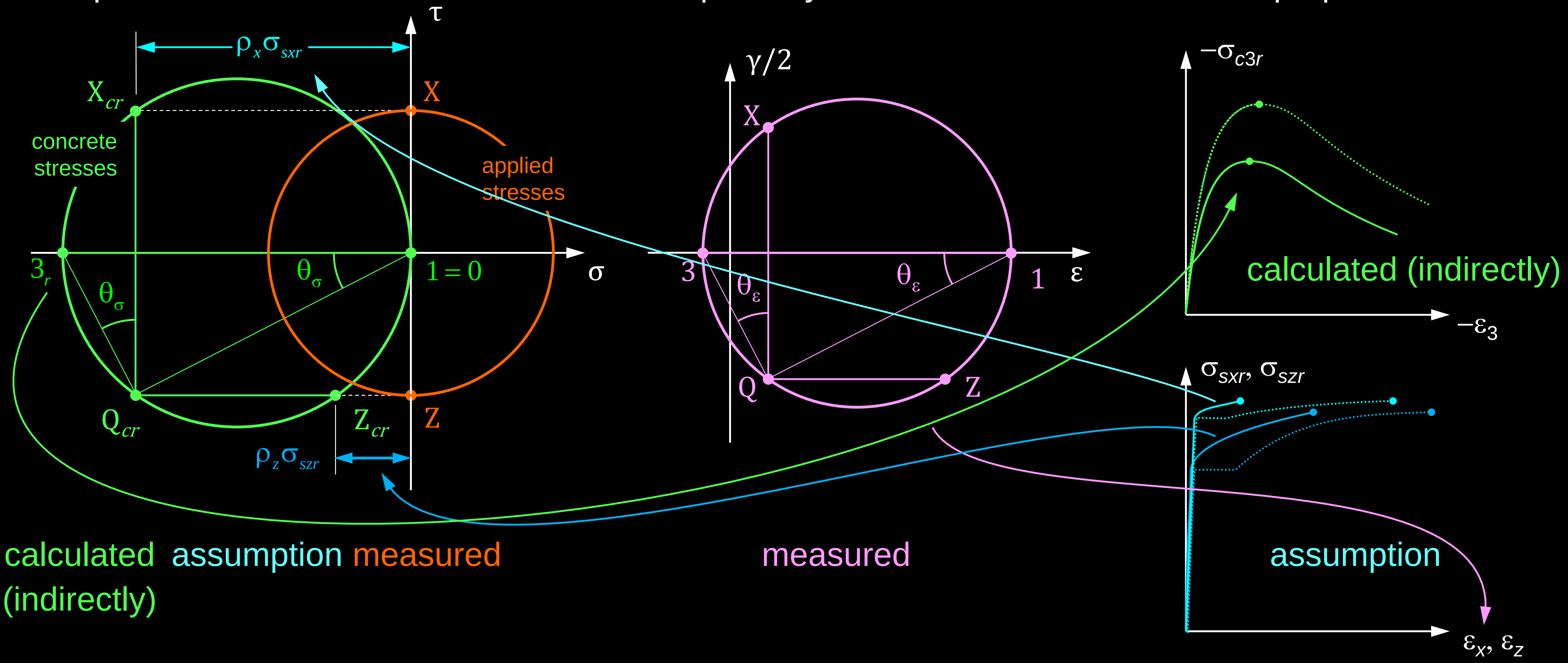
Material properties



Measured or calculated?

Determination of stress and strain state in experiments with conventional measurements

Equilibrium Compatibility Material properties



Measured or calculated?

Determination of stress and strain state in experiments with continuous strain measurement (fibres)