2 In plane loading – walls and beams

2.3 Compatibility and deformation capacity

Learning objectives

Within this chapter, the students are able to:

- determine the behaviour of concrete as a function of the compressive strength and the cracking state.
- recognise the assumption of limit analysis methods for the materials having sufficient deformation capacity to reach the plastic solution without rupturing, and the existence of approaches to verify the deformation capacity of the materials.
- evaluate plastic redistributions of internal forces in hyperstatic systems (beams and frames) subjected either to external loads or imposed deformations, and calculate the deformation demand in elements subjected to bending or normal actions.
- estimate the deformation capacity of a structure subjected to bending or normal actions.
  - explain the tension-stiffening effect and how it affects the structural behaviour.
  - illustrate the main assumptions and behaviour of bonded reinforcement according to the Tension Chord Model.

2 In plane loading – walls and beams

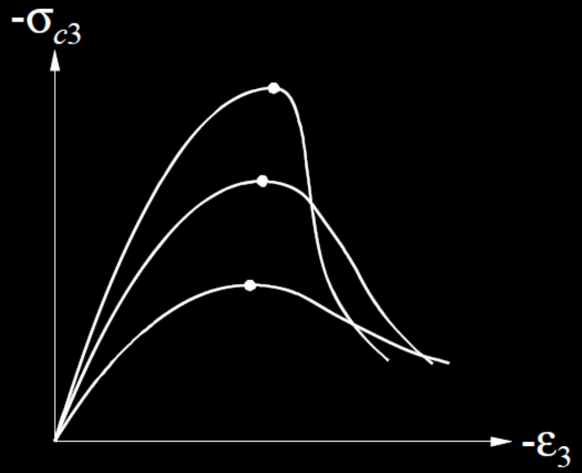
2.3 Compatibility and deformation capacity

A) Behaviour of concrete in compression

Behaviour of concrete in compression

Main factors influencing the equivalent strength to be considered in plastic calculations

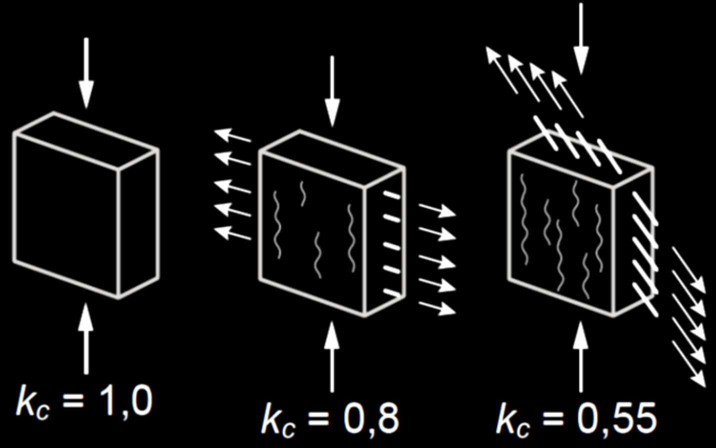
Strain softening after peak strength (material effect)



The concrete brittleness (i.e. relative amount of softening) increases with the compressive strength and also the reduction of the strength to be accounted for (η_{fc}).

$$f_{cd} = \frac{\eta_{f_c} f_{ck}}{\gamma_c} \qquad \eta_{f_c} = \left(\frac{30}{f_{ck}}\right)^{1/3} \leq 1{,}0$$

Influence of transverse cracking on concrete strength (structural effect)



Reduction factor to account for this effect (k_c) can be determined in a more refined manner based on the state of deformations (see following slides).

Equivalent strength

$$k_c f_{cd} = k_c \frac{\eta_{f_c} f_{ck}}{\gamma_c}$$

Behaviour of concrete in compression

Dependence of the concrete compressive strength and shear resistance on the strain state

Tests have shown that the compressive strength in membrane elements is reduced by (imposed) transverse strains.

In 1986, Vecchio and Collins proposed reducing the compressive strength by a factor $1/(0.8+170\cdot\varepsilon_1)\leq 1$ (assuming "average" concrete stresses).
This also takes implicitly other effects into account.

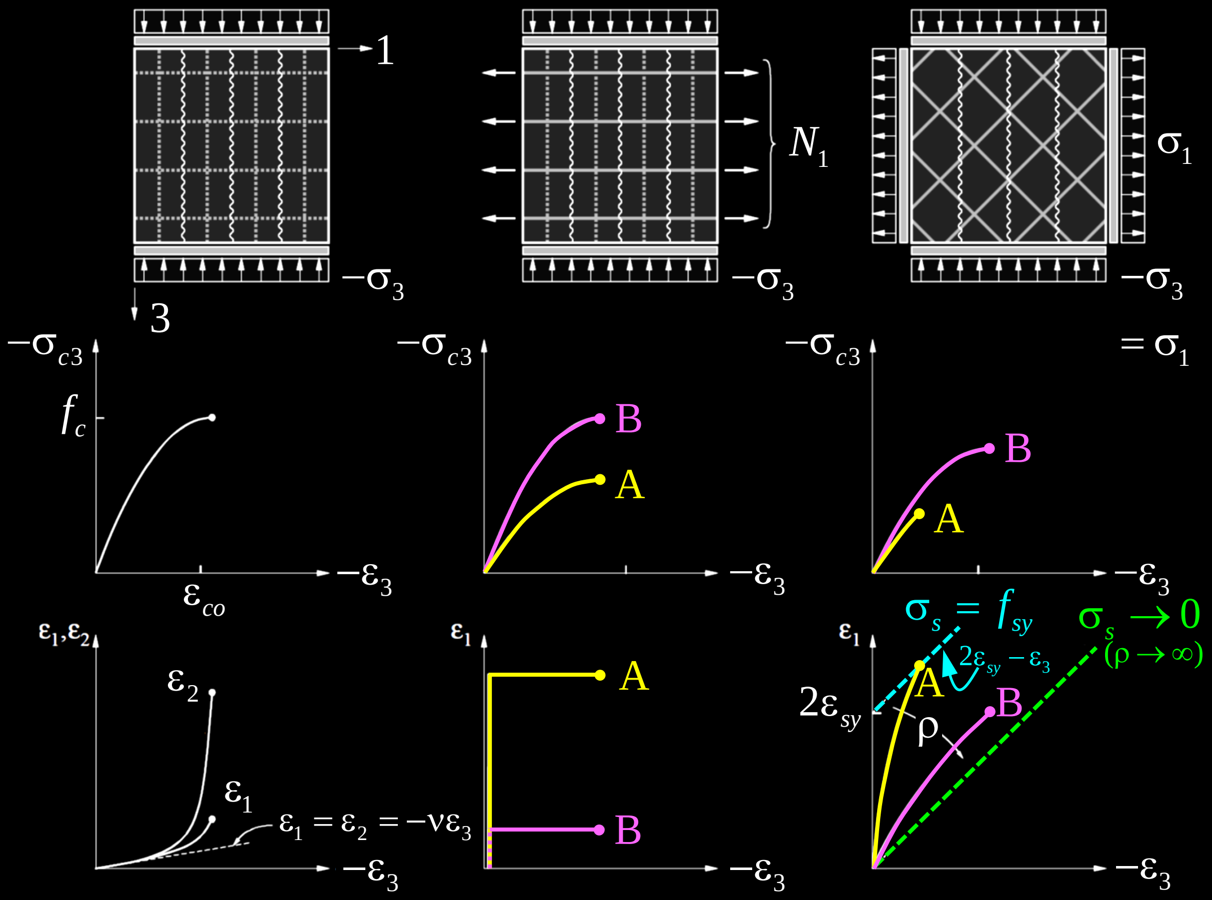
In 1998, Kaufmann proposed to consider additionally the (already known) inversely proportional increase of the compressive strength with the cylinder compressive strength:

$$f_{ce}=\frac{f_{c,cyl}^{2/3}}{0.4+30\cdot\varepsilon_1}$$

On the basis of this and other work, SIA 262 has introduced the following coefficient for the verification of webs of beams:

$$k_c=\frac{1}{1,2+55\varepsilon_1}\leq 0.65$$

This can be applied in a more general way to any structural member when removing the 0.65 upper-bound

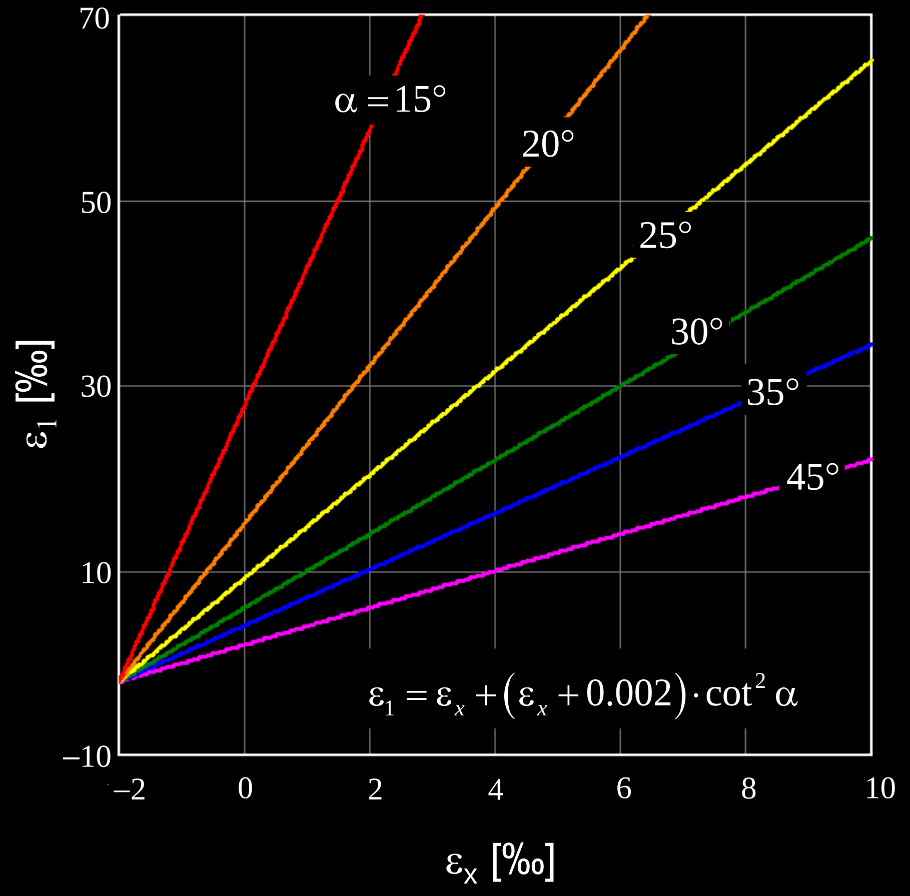


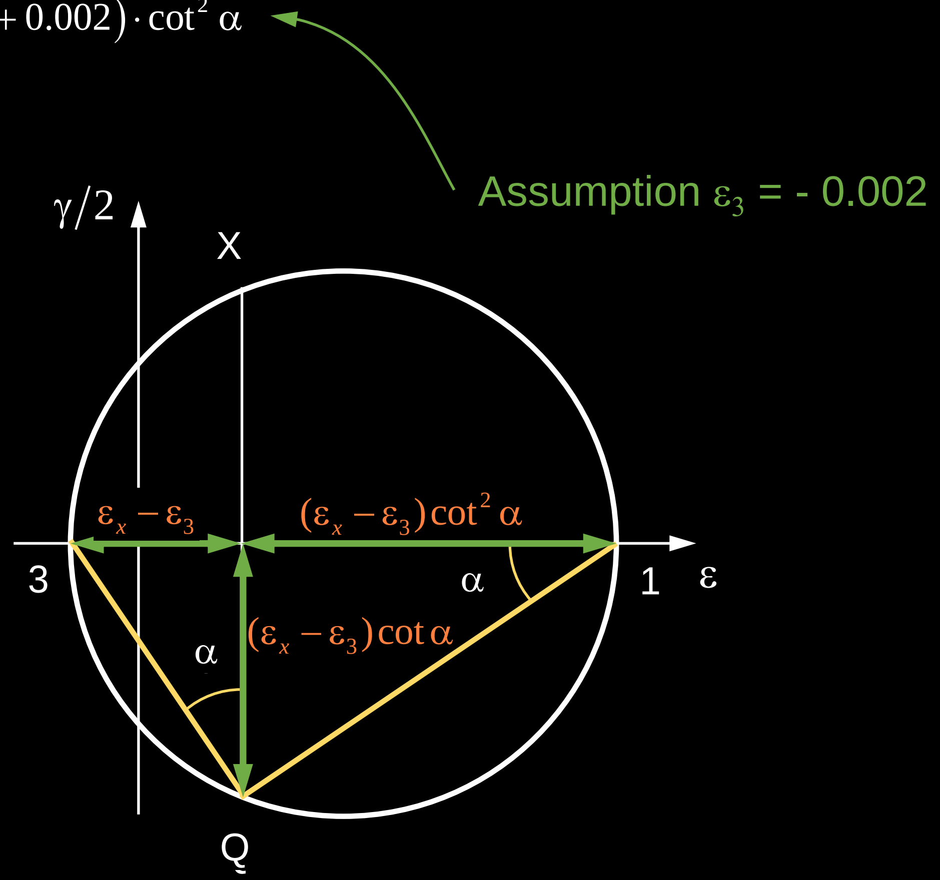
Behaviour of concrete in compression

Concrete compressive strength and shear resistance as a function of the strain state

$$k_c(\varepsilon_x, \alpha) = \frac{1}{1.2 + 55 \cdot \varepsilon_1} \leq 0.65$$

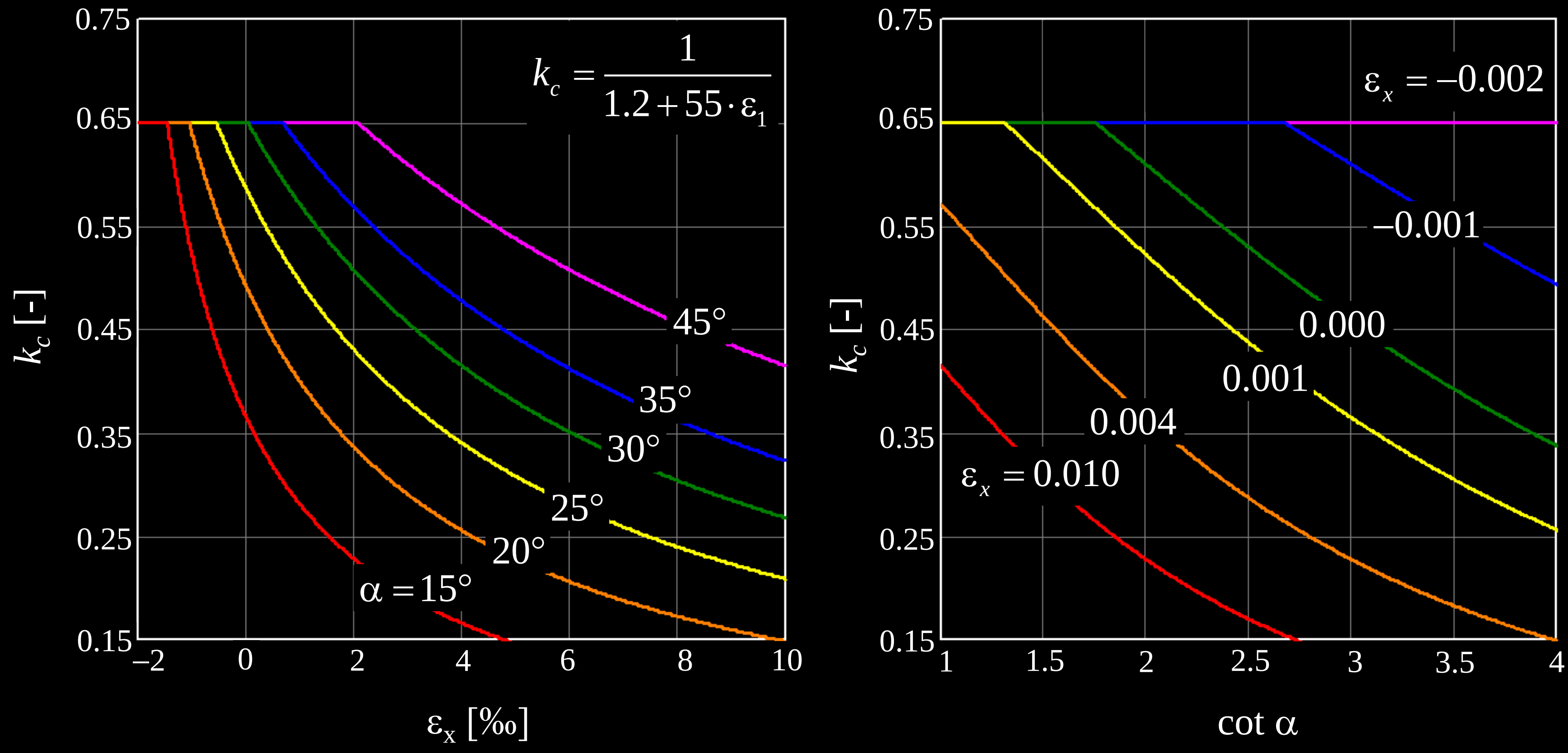
$$\varepsilon_1 = \varepsilon_x + \left(\varepsilon_x + 0.002\right) \cdot \cot^2 \alpha$$





Behaviour of concrete in compression

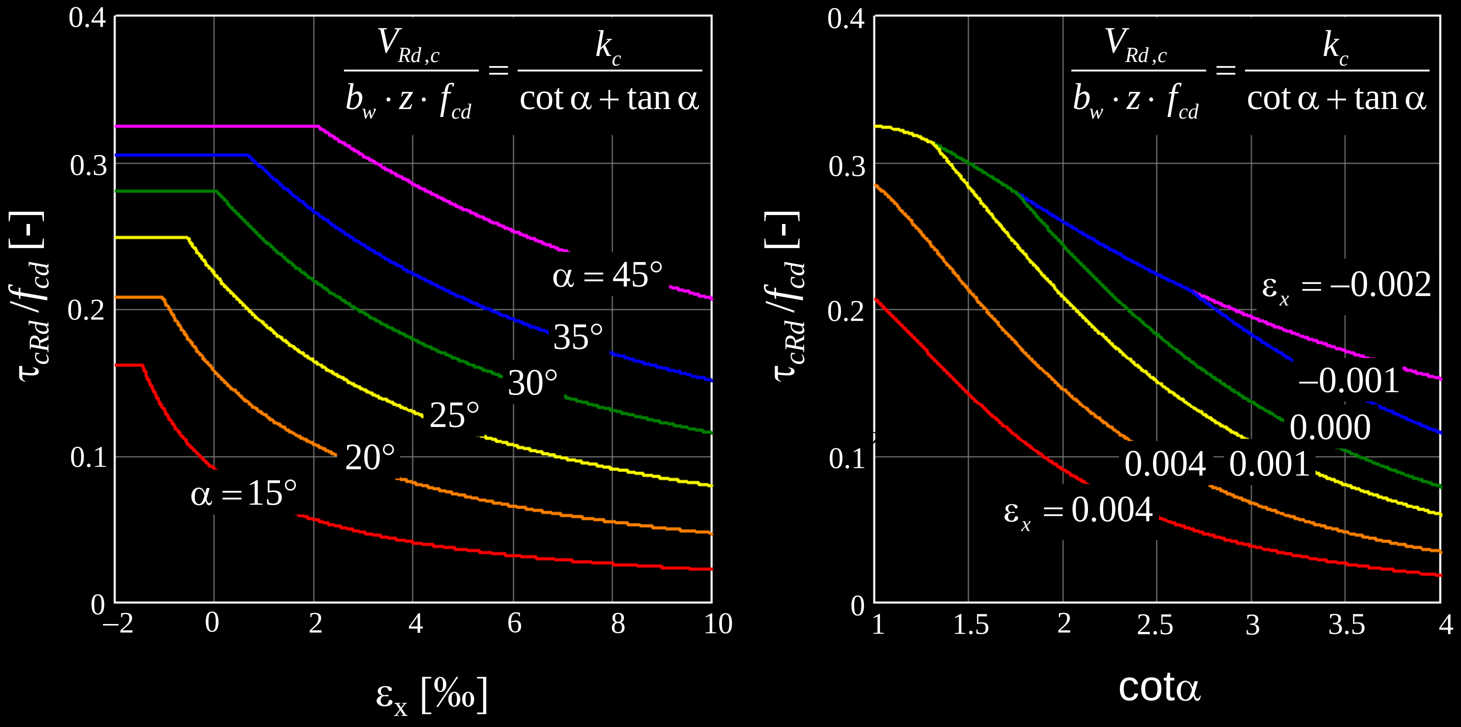
Concrete compressive strength and shear resistance as a function of the strain state



→ $k_c \cdot f_c$ is largely reduced for flat inclinations of the compression field and for plastic strains of the tension chord

Behaviour of concrete in compression

Concrete compressive strength and shear resistance as a function of the strain state



$$\frac{1}{\cot\alpha + \tan\alpha} = \sin\alpha \cdot \cos\alpha$$

→ $k_c \cdot f_c$ is largely reduced for flat inclinations of the compression field and for plastic strains of the tension chord

→ Concrete compressive stresses increase sharply with flat inclinations (see above)

→ Very flat inclinations do not make sense when dimensioning, but are often necessary when assessing old bridges

→ Attention to plastic internal force redistributions from the support (= large shear force)

2 In plane loading – walls and beams

2.3 Compatibility and deformation capacity

B) Behaviour of bonded reinforcement

Behaviour of the bonded reinforcement – Tension chord model (SBI)

Differential equations of bond

General bond-slip law

Equilibrium of an element with length dx:

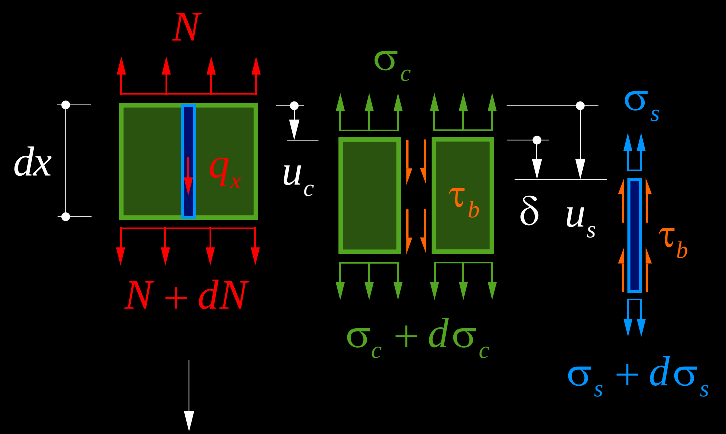
$$\frac{d\sigma_c}{dx} = -\frac{\varnothing\pi\tau_b + q_x}{A_c(1-\rho)}; \qquad \frac{d\sigma_s}{dx} = \frac{4\tau_b}{\varnothing}$$

→ ODE of 1st order

Considering linear elastic material behaviour:

$$\frac{d^2\delta}{dx^2} = \frac{4\tau_b}{\varnothing E_s} + \frac{\varnothing\pi\tau_b + q_x}{A_c E_c(1-\rho)}$$

→ ODE of 2nd order



Simplified bond-slip law, used in TCM

Equilibrium of an element with length dx:

$$\frac{d\sigma_c}{dx} = -\frac{\varnothing\pi\tau_b + q_x}{A_c(1-\rho)} = \text{const*}; \qquad \frac{d\sigma_s}{dx} = \frac{4\tau_b}{\varnothing} = \text{const*}$$

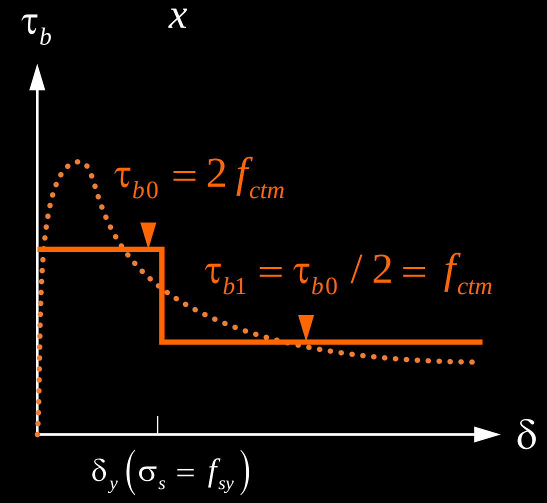
→ linear (integrate once)

Considering linear elastic material behaviour:

$$\frac{d^2\delta}{dx^2} = \frac{4\tau_b}{\varnothing E_s} + \frac{\varnothing\pi\tau_b + q_x}{A_c E_c(1-\rho)} = \text{const*}$$

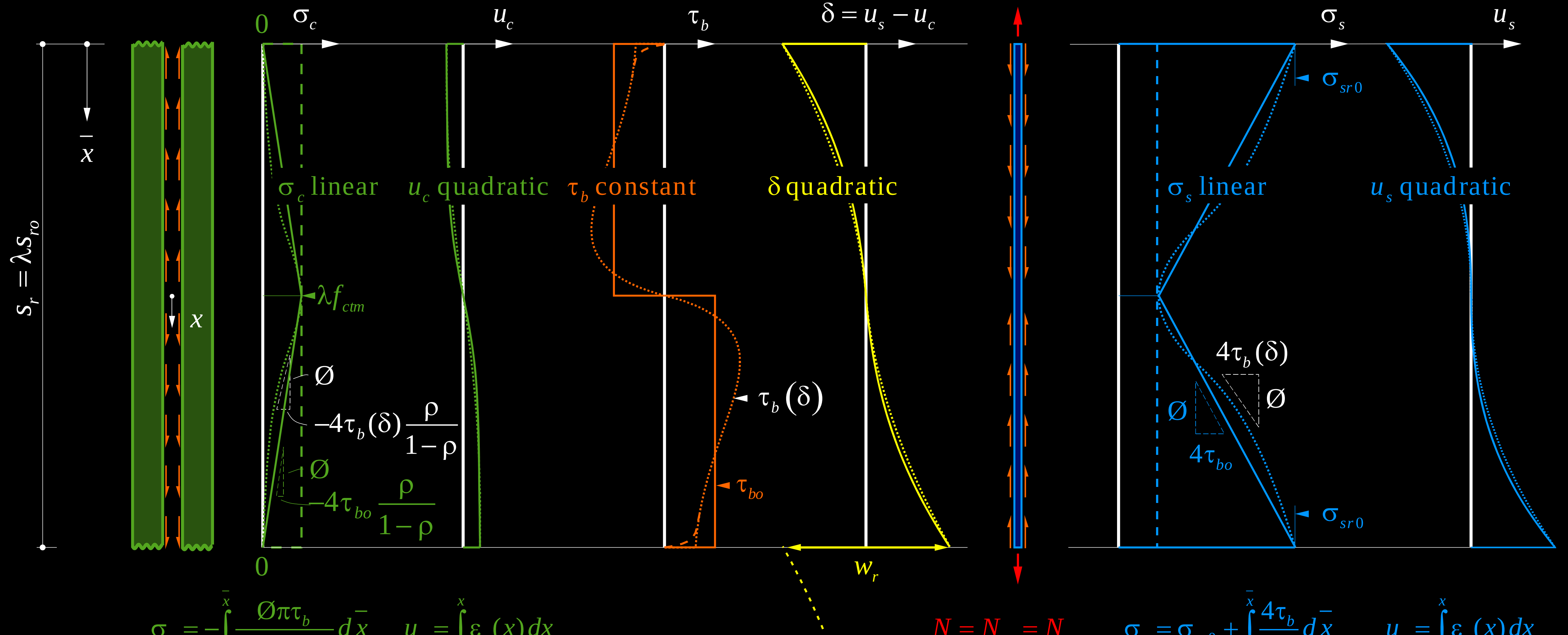
→ quadratic (integrate twice)

* if q_x = const



Behaviour of the bonded reinforcement – Tension chord model (SBI)

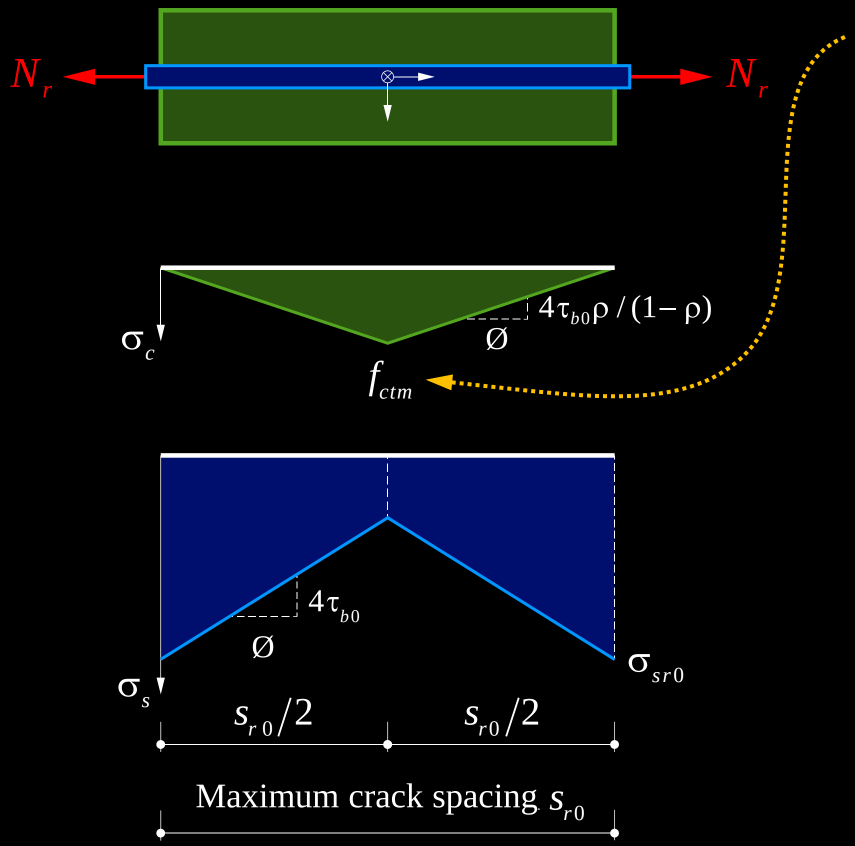
Crack element at crack formation $N = N_r$



$\sigma_c = -\int_0^{\bar{x}} \frac{Ø\pi\tau_b}{A_c(1-\rho)} d\bar{x}$ $\quad u_c = \int_0^{x} \varepsilon_c(x)\,dx$ $\quad N = N_{sr} = N_r$ $\quad \sigma_s = \sigma_{sr0} + \int_0^{\bar{x}} \frac{4\tau_b}{Ø} d\bar{x}$ $\quad u_s = \int_0^{x} \varepsilon_s(x)\,dx$

Behaviour of the bonded reinforcement – Tension chord model (SBI)

View of a tension chord (total cross-section A_c), reinforced with bar with diameter Ø ([6], page 3.5f)



Concrete stress in the middle of the element with length s_{r0} (maximum crack spacing) is $\sigma_c = f_{ctm}$ i.e. another crack could form there.

$$s_{rm0} \approx \frac{\varnothing}{4} \cdot \left(\frac{1}{\rho_t} - 1\right)$$

Thus the minimum crack spacing is:

$$s_{r,min} = s_{r0}/2$$

Generally, the crack spacing varies with parameter λ:

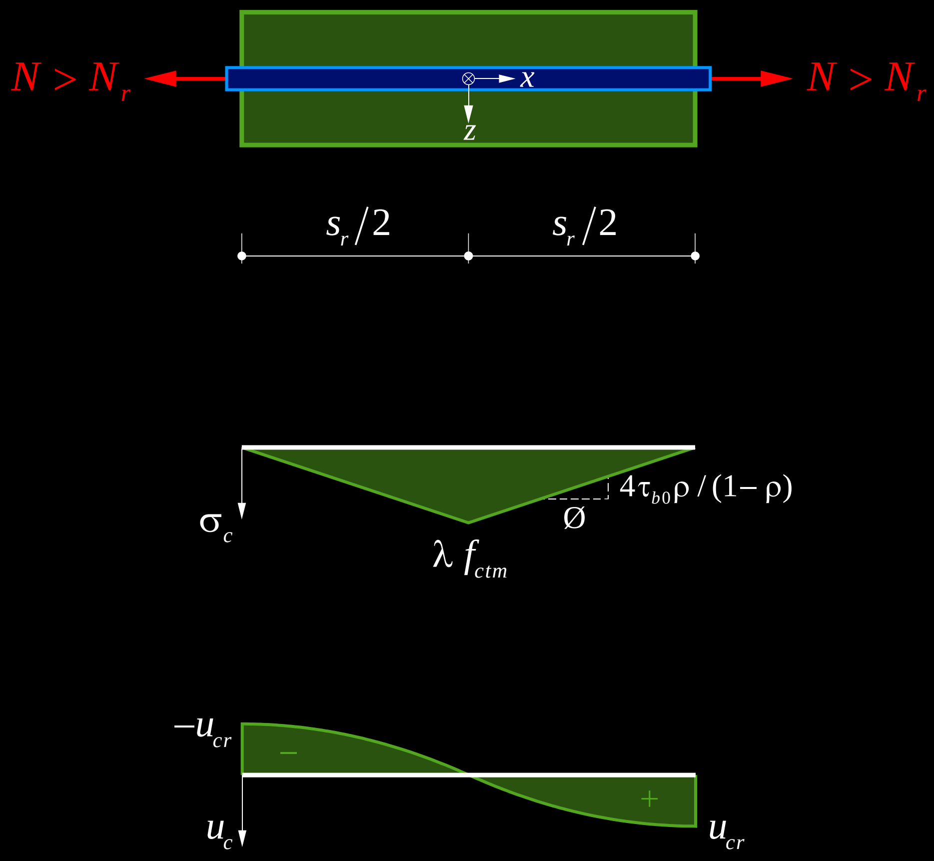
$$s_r = \lambda \cdot s_{r0} \qquad \left(\frac{1}{2} < \lambda < 1\right)$$

→ Theoretical limits of the crack spacing with fully developed crack pattern!

SN: If the cracks form because of applied loads, the fully developed crack pattern forms at once (theoretically).

Behaviour of the bonded reinforcement – Tension chord model (SBI)

Increase of normal force after crack formation $N > N_r$ ([6], page 3.5f)



Concrete stresses remain constant after cracking.

Steel stresses keep increasing.

Mean concrete elongation

$$\varepsilon_{cm} = \frac{\int_0^{s_r} \varepsilon_c \, dx}{s_r} = \frac{\int_0^{s_r} \frac{\sigma_c}{E_c} \, dx}{s_r} = \frac{\lambda f_{ctm}}{2E_c}$$

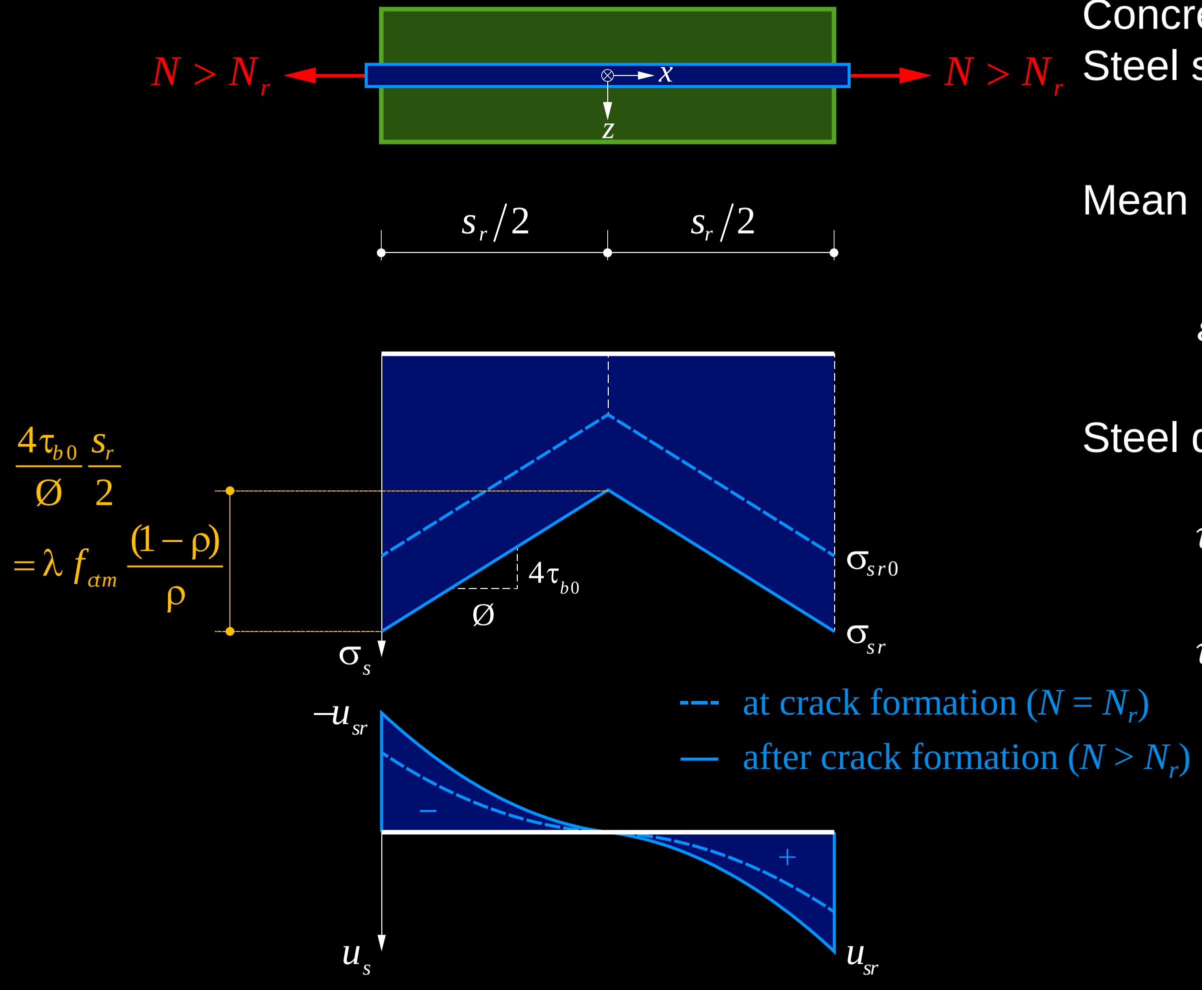
Concrete displacements

$$u_c(x) = \int_0^x \varepsilon_c(x) \, dx = \int_0^x \frac{\sigma_c(x)}{E_c} \, dx$$

$$u_{cr} = u_c\left(x = \frac{s_r}{2}\right)$$

Behaviour of the bonded reinforcement – Tension chord model (SBI)

Increase of normal force after crack formation $N > N_r$ ([6], page 3.5f)



Concrete stresses remain constant after cracking.
Steel stresses keep increasing.

Mean steel elongation

$$\varepsilon_{sm} = \frac{\int_0^{s_r} \frac{\sigma_s}{E_s}\, dx}{s_r} = \frac{\sigma_{sr}}{E_s} - \frac{4\tau_{b0}}{\emptyset}\frac{s_r}{4E_s} = \frac{\sigma_{sr}}{E_s} - \frac{\lambda f_{ctm}(1-\rho)}{2\rho E_s}$$

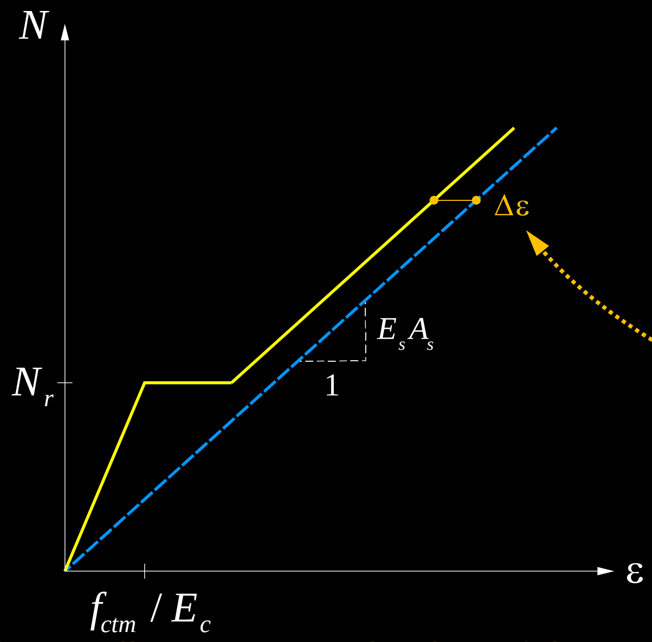
Steel displacements

$$u_s(x) = \int_0^x \varepsilon_s(x)\, dx = \int_0^x \frac{\sigma_s(x)}{E_s}\, dx$$

$$u_{sr} = u_s\left(x = \frac{s_r}{2}\right)$$

Behaviour of the bonded reinforcement – Tension chord model (SBI)

Increase of normal force after crack formation $N > N_r$ ([6], page 3.5f)



N-ε- and σ_{sr}-ε-diagrams : Reduction of the elongation of the bare steel by Δε
(Δε remains constant until yielding).

NB: Good approximation for w_r (small ρ)

$$\frac{\varnothing/4\rho}{2E_s}\left(\frac{N}{A_s}-\frac{f_{ctm}}{4\rho}\right) \leq w_r \leq \frac{\varnothing/4\rho}{E_s}\left(\frac{N}{A_s}-\frac{f_{ctm}}{2\rho}\right)$$

Concrete stresses remain constant after cracking. Steel stresses keep increasing.

Steel elongation at crack

$$\varepsilon_{sr} = \sigma_{sr}/E_s$$

Average concrete elongation

$$\varepsilon_{cm} = \lambda f_{ctm}/(2E_c)$$

Mean steel elongation

$$\varepsilon_{sm} = \frac{\sigma_{sr}}{E_s} - \frac{\tau_{b0}}{\varnothing}\frac{s_r}{E_s} = \frac{\sigma_{sr}}{E_s} - \frac{\lambda f_{ctm}(1-\rho)}{2\rho E_s}$$

Crack widths: Difference of the mean steel and concrete strains multiplied by s_r (λ = 0.5...1):

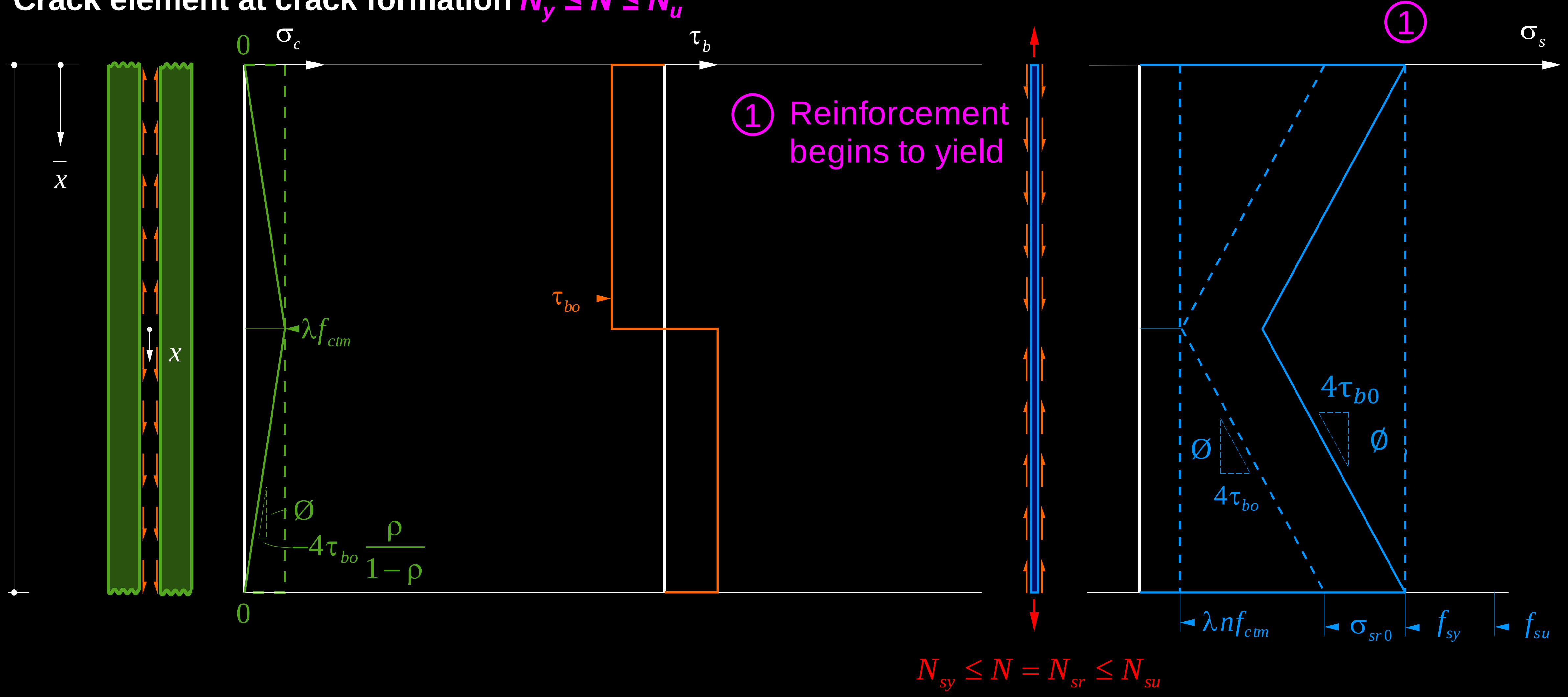
$$w_r = s_r\left[\frac{\sigma_{sr}}{E_s} - \frac{\lambda f_{ctm}(1-\rho)}{2\rho E_s} - \frac{\lambda f_{ctm}}{2E_c}\right] = \frac{\lambda s_{r0}(2\sigma_{sr} - \lambda\sigma_{sr0})}{2E_s}$$

with $\sigma_{sr} = N/A_s$

$$\frac{s_{r0}}{2E_s}\left(\sigma_{sr} - \frac{\sigma_{sr0}}{4}\right) \leq w_r \leq \frac{s_{r0}}{E_s}\left(\sigma_{sr} - \frac{\sigma_{sr0}}{2}\right)$$

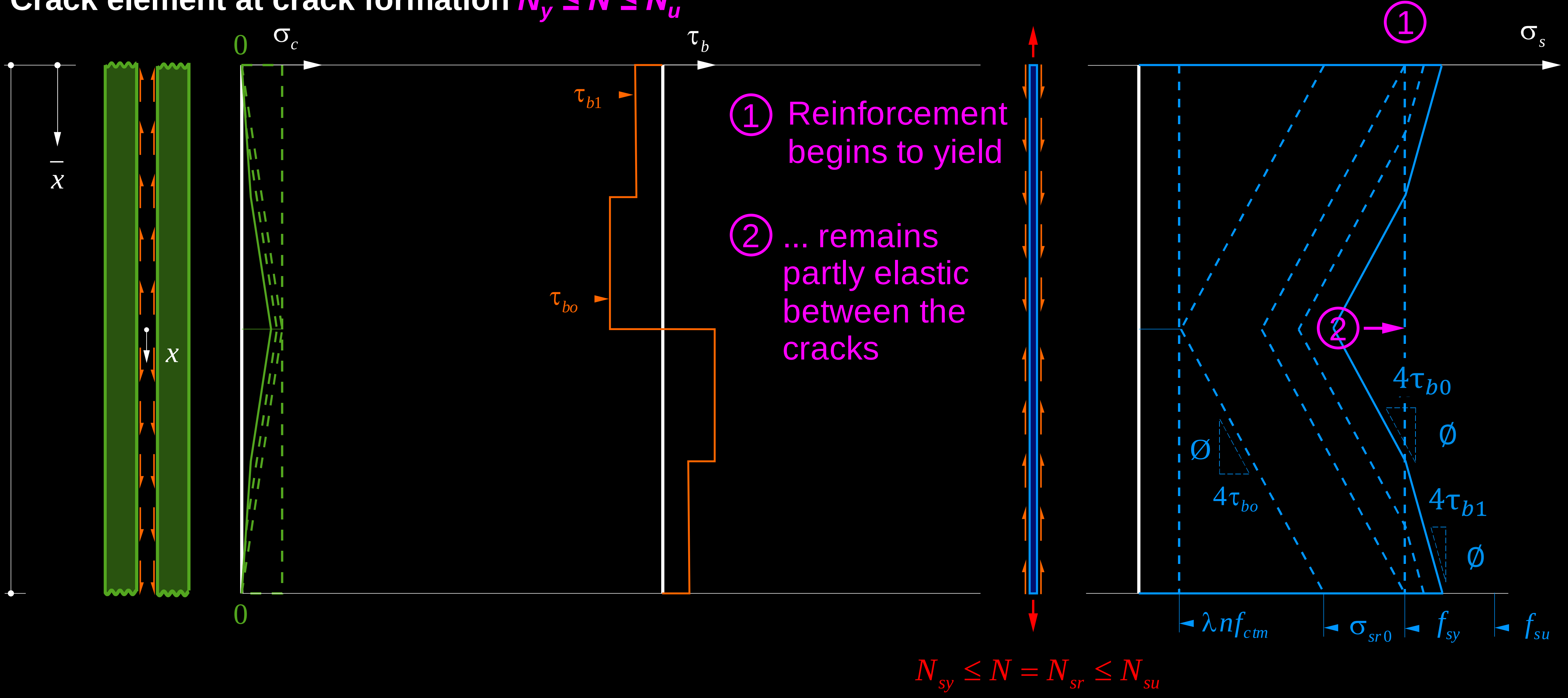
Behaviour of the bonded reinforcement – Tension chord model (SBI)

Crack element at crack formation $N_y \leq N \leq N_u$



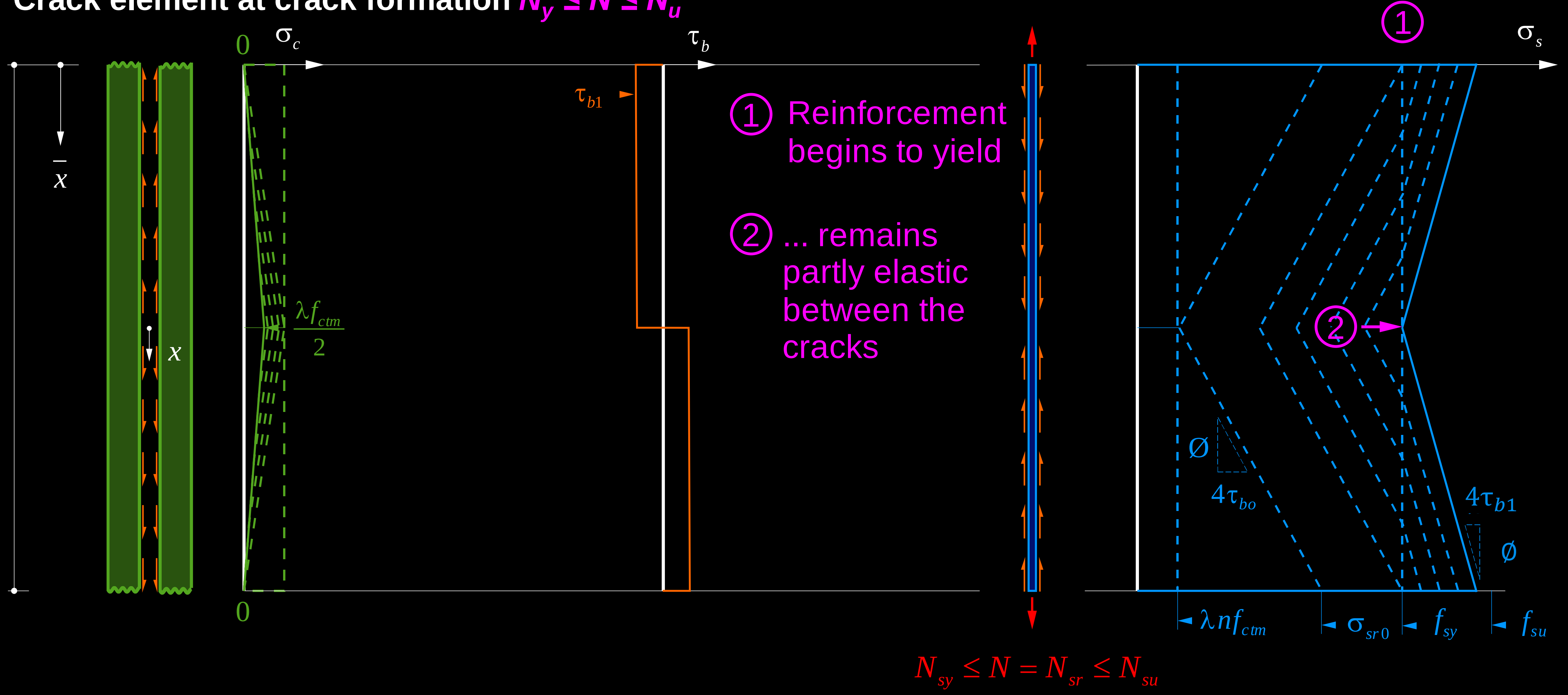
Behaviour of the bonded reinforcement – Tension chord model (SBI)

Crack element at crack formation $N_y \leq N \leq N_u$



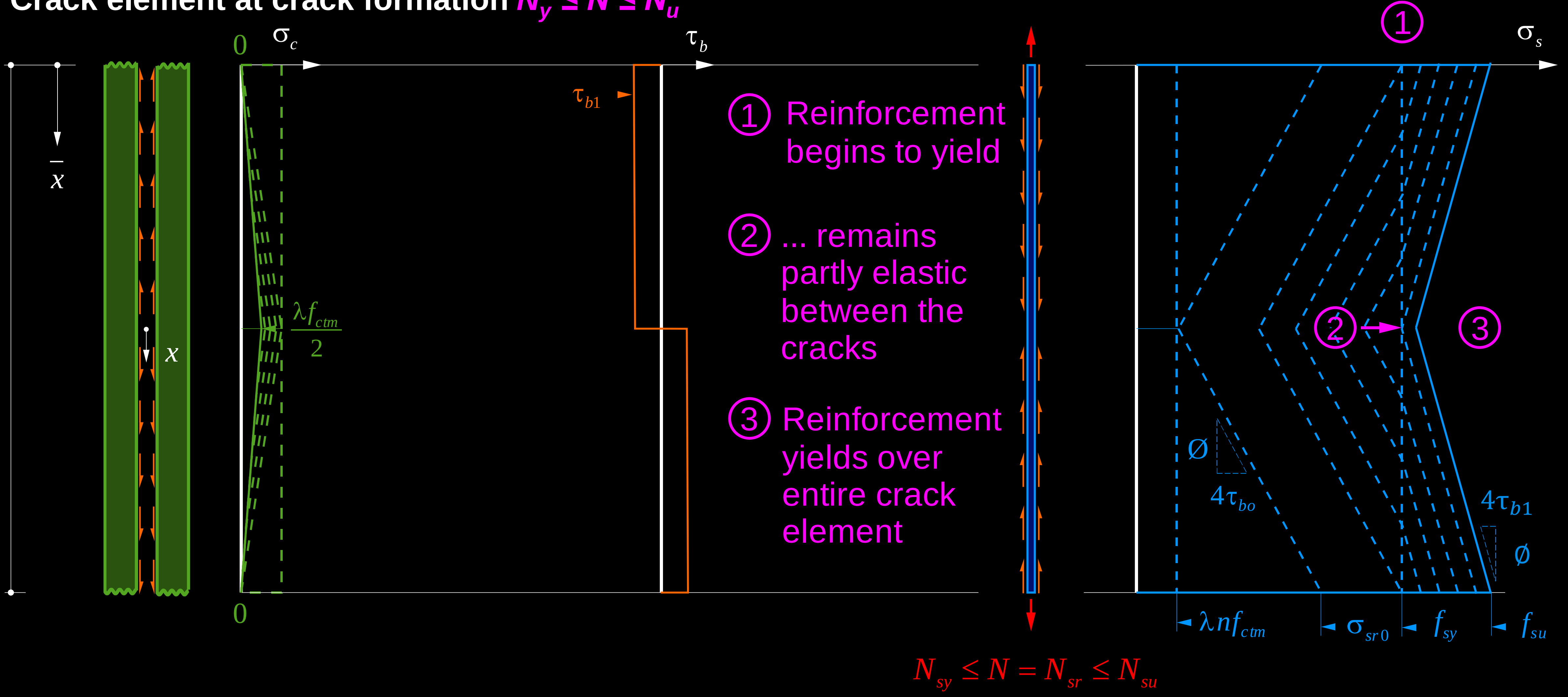
Behaviour of the bonded reinforcement – Tension chord model (SBI)

Crack element at crack formation $N_y \leq N \leq N_u$



Behaviour of the bonded reinforcement – Tension chord model (SBI)

Crack element at crack formation $N_y \leq N \leq N_u$



Behaviour of the bonded reinforcement – Tension chord model (SBI)

Closed form solution for a bilinear steel stress-strain relationship

① Elastic reinforcement over entire crack element

$\sigma_{sr} \leq f_{sy}$

$$\sigma_{sr} = E_s \varepsilon_{sm} + \frac{\tau_{b0} s_r}{\varnothing}$$

$$\varepsilon_{sm} = \frac{\sigma_{sr}}{E_s} - \frac{\tau_{b0} s_r}{E_s \varnothing} = \frac{\sigma_{sr}}{E_s} - \lambda \frac{f_{ctm}(1-\rho)}{2E_s\rho}$$

"bare steel – $\Delta\varepsilon_0$"

$$\Delta\varepsilon_0 = \frac{\tau_{b0} s_r}{E_s \varnothing} = \lambda \frac{f_{ctm}(1-\rho)}{2E_s\rho}$$

② Reinforcement yields near cracks, elastic between cracks

$f_{sy} \leq \sigma_{sr} \leq \left(f_{sy} + \frac{2\tau_{b1} s_r}{\varnothing}\right)$

$$\sigma_{sr} = f_{sy} + 2\frac{\frac{\tau_{b0} s_r}{\varnothing} - \sqrt{\left(f_{sy} - E_s \varepsilon_{sm}\right)\frac{\tau_{b1} s_r}{\varnothing}\left(\frac{\tau_{b0}}{\tau_{b1}} - \frac{E_s}{E_{sh}}\right) + \frac{E_s}{E_{sh}} \tau_{b0}\tau_{b1}\frac{s_r^2}{\varnothing^2}}}{\left(\frac{\tau_{b0}}{\tau_{b1}} - \frac{E_s}{E_{sh}}\right)}$$

$$\varepsilon_{sm} = \frac{\left(\sigma_{sr} - f_{sy}\right)^2 \varnothing}{4E_{sh}\tau_{b1} s_r}\left(1 - \frac{E_{sh}\tau_{b0}}{E_s \tau_{b1}}\right) + \frac{\left(\sigma_{sr} - f_{sy}\right)}{E_s}\frac{\tau_{b0}}{\tau_{b1}} + \left(\varepsilon_{sy} - \frac{\tau_{b0} s_r}{E_s \varnothing}\right)$$

③ Reinforcement yields over entire crack element

$\left(f_{sy} + \frac{2\tau_{b1} s_r}{\varnothing}\right) \leq \sigma_{sr} \leq f_{su}$

$$\sigma_{sr} = f_{sy} + E_{sh}\left(\varepsilon_{sm} - \frac{f_{sy}}{E_s}\right) + \frac{\tau_{b1} s_r}{\varnothing}$$

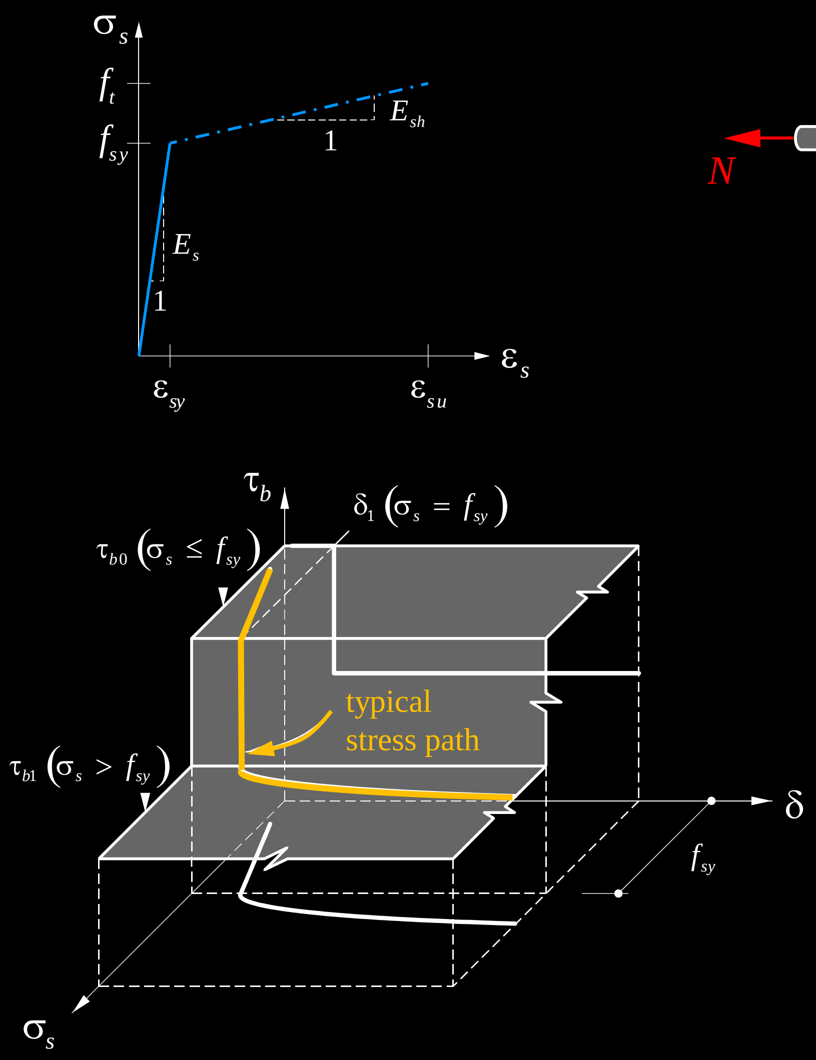
$$\varepsilon_{sm} = \varepsilon_{sy} + \frac{\left(\sigma_{sr} - f_{sy}\right)}{E_{sh}} - \frac{\tau_{b1} s_r}{E_{sh}\varnothing}$$

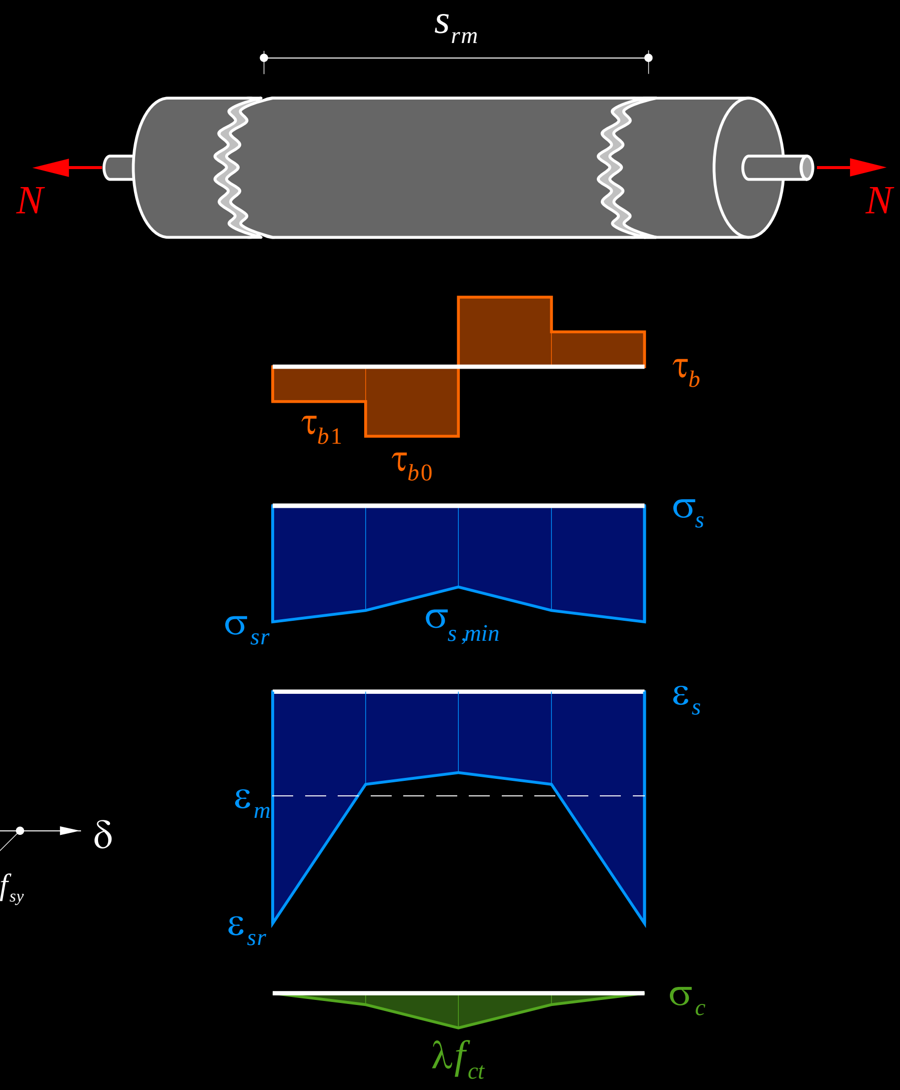
"bare steel – $\Delta\varepsilon_1$"

$$\Delta\varepsilon_1 = \frac{\tau_{b1} s_r}{E_{sh}\varnothing}$$

Behaviour of the bonded reinforcement – Tension chord model (SBI)

Constitutive relationship of the bonded reinforcement (tension chord model with bilinear bare reinforcement):



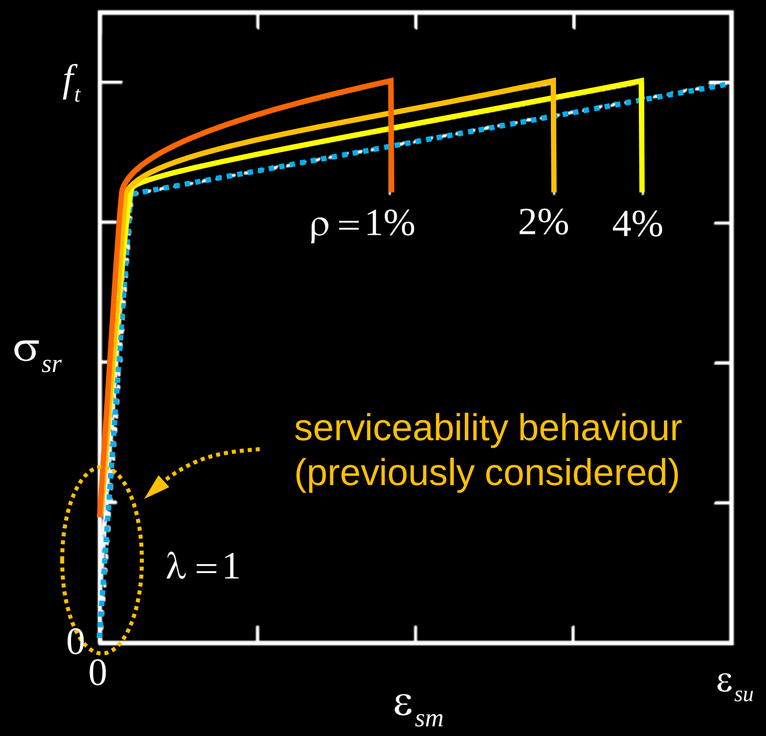


Behaviour of the bonded reinforcement – Tension chord model (SBI)

Load-deformation behaviour considering bond (influence at high loads)

→ No influence on tensile resistance

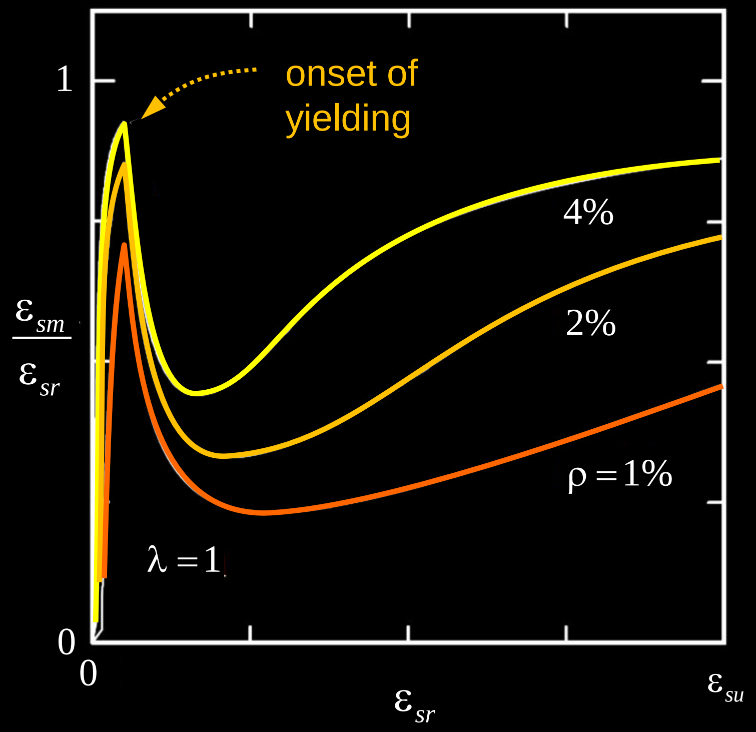
→ Stiffer behaviour than bare steel



Ratio of average elongation to maximum elongation at the cracks considering bond

→ Heavy drop after onset of yielding

→ Pronounced influence on ductility!



$f_s = 500\,\text{MPa}$

$f_{su} = 625\,\text{MPa}$

$E_s = 200\,\text{GPa}$

$\varepsilon_{su} = 0.05$

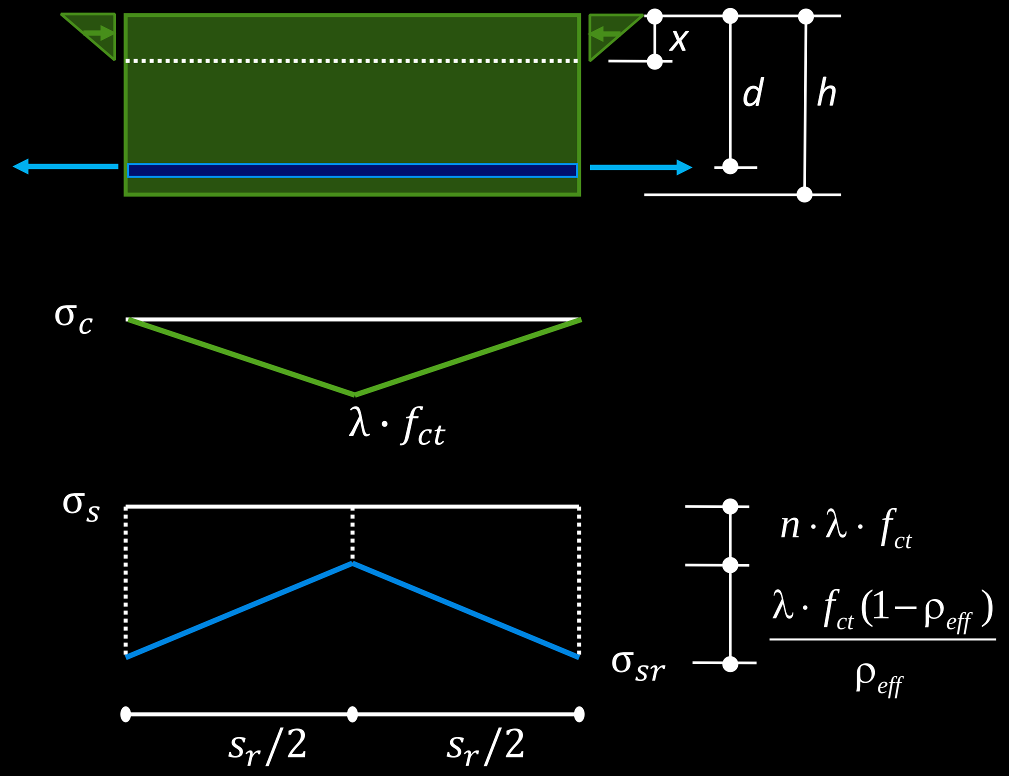
$\varnothing = 16\text{ mm}$

$f_c = 30\text{ MPa}$

Behaviour of the bonded reinforcement

Application to loading cases different than uniaxial tension

Simple bending (SB I): Elastic bending stiffness – tensile stiffness [6], page 2.16f



Setting the steel stress at the crack at the onset of cracking ($M = M_r$)

$$\sigma_{sr0} = \frac{M_r}{A_s(d-x/3)} = \frac{M_r(d-x)E_s}{EI^{II}} \quad \text{mit } EI^{II} = A_s E_s (d-x)(d-x/3)$$

equal to the steel stress at cracking of a tension chord

$$\sigma_{sr0} = f_{ct}\left(\frac{1}{\rho_{eff}} + n - 1\right)$$

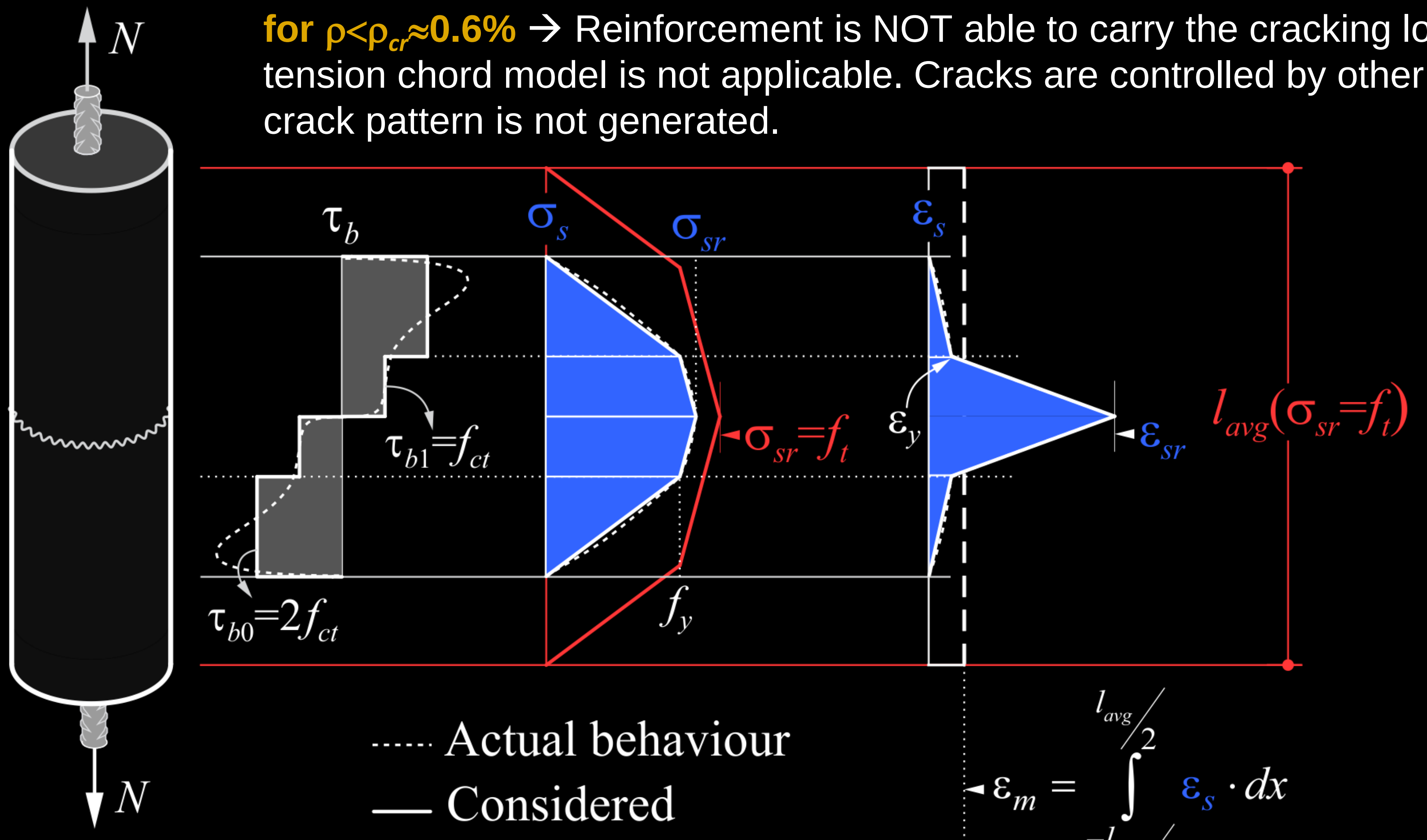
one obtains the equivalent reinforcement ratio ρ_{eff} :

$$\rho_{eff} = \frac{1}{\frac{M_r(d-x)E_s}{f_{ct} EI^{II}} + 1 - n}$$

Behaviour of the bonded reinforcement

Tension stiffening for non-stabilised crack patterns (pull-out model)

for $\rho<\rho_{cr}\approx 0.6\%$ → Reinforcement is NOT able to carry the cracking load without yielding and the tension chord model is not applicable. Cracks are controlled by other reinforcement and a stabilized crack pattern is not generated.



$$\varepsilon_m = \int_{-l_{avg}/2}^{l_{avg}/2} \varepsilon_s \cdot dx$$

- A pull-out tension stiffening model can be formulated for these situations by assuming (a) independent cracks and (b) the same bond slip model as for the tension chord model.
- A certain crack spacing (l_{avg}) should be assumed to compute the average reinforcement strain.

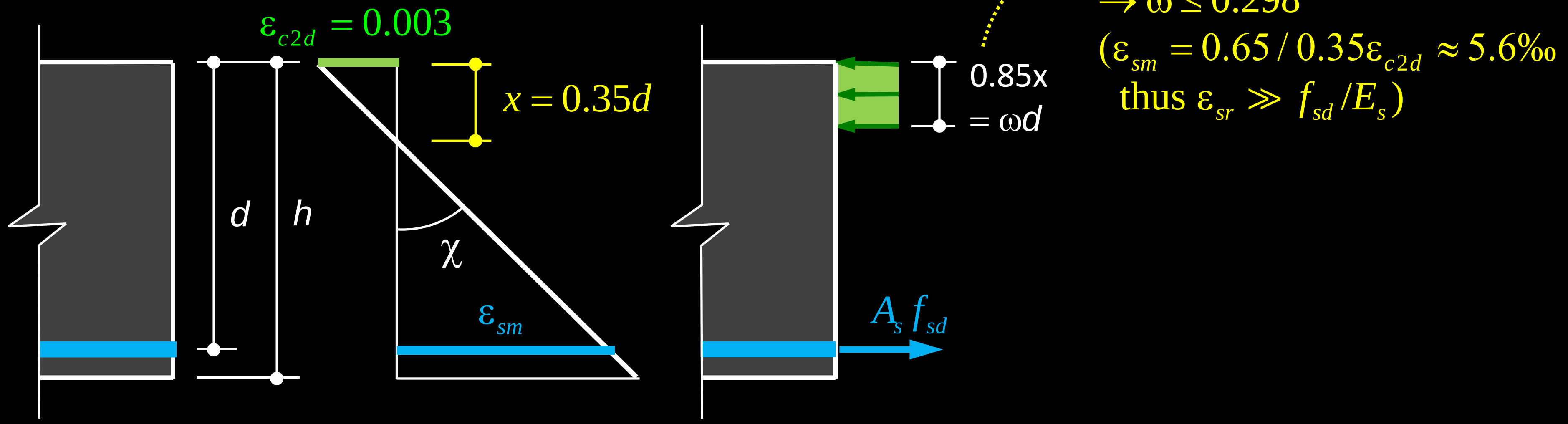
2 In plane loading – walls and beams

2.3 Compatibility and deformation capacity

C) Deformation capacity of beams

Beams – Deformation capacity

Limitation of the compression zone depth according to SIA 262



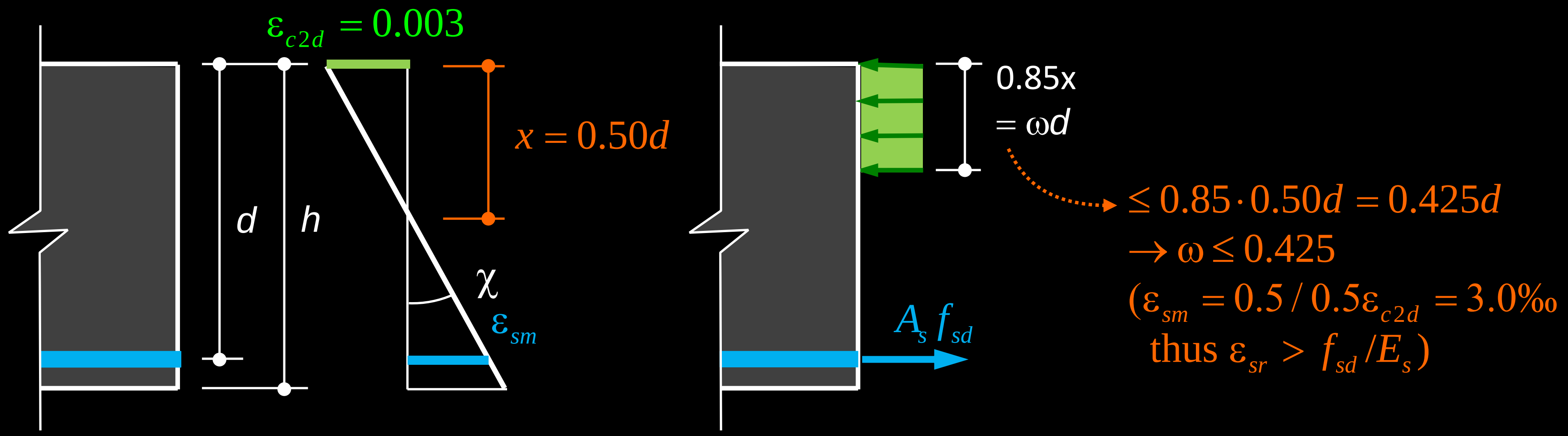
Maximum reinforcement ratio and bending resistance according to SIA 262, section 4.1.4.2:
(for components mainly subjected to bending)

- x/d ≤ 0.35: Internal force redistributions **without** verification of deformation capacity

$$x/d \leq 0.35 \rightarrow \omega \leq 0.298 \rightarrow M_{Rd} \leq bd^2 f_{cd} \omega \cdot (1-\omega/2) = 0.253 \cdot bd^2 f_{cd}$$

Beams – Deformation capacity

Limitation of the compression zone depth according to SIA 262



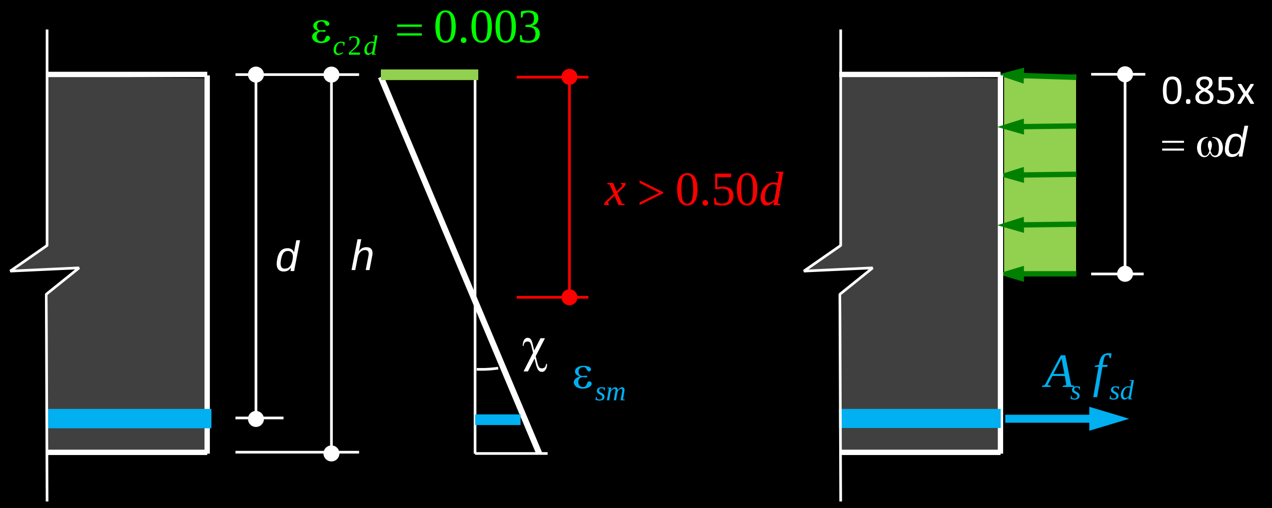
Maximum reinforcement ratio and bending resistance according to SIA 262, section 4.1.4.2:
(for components mainly subjected to bending)

- 0.35 ≤ x/d ≤ 0.5: Internal force redistributions **with** verification of deformation capacity

$$x/d \leq 0.50 \rightarrow \omega \leq 0.425 \rightarrow M_{Rd} \leq bd^2 f_{cd} \omega \cdot (1-\omega/2) = 0.335 \cdot bd^2 f_{cd}$$

Beams – Deformation capacity

Limitation of the compression zone depth according to SIA 262

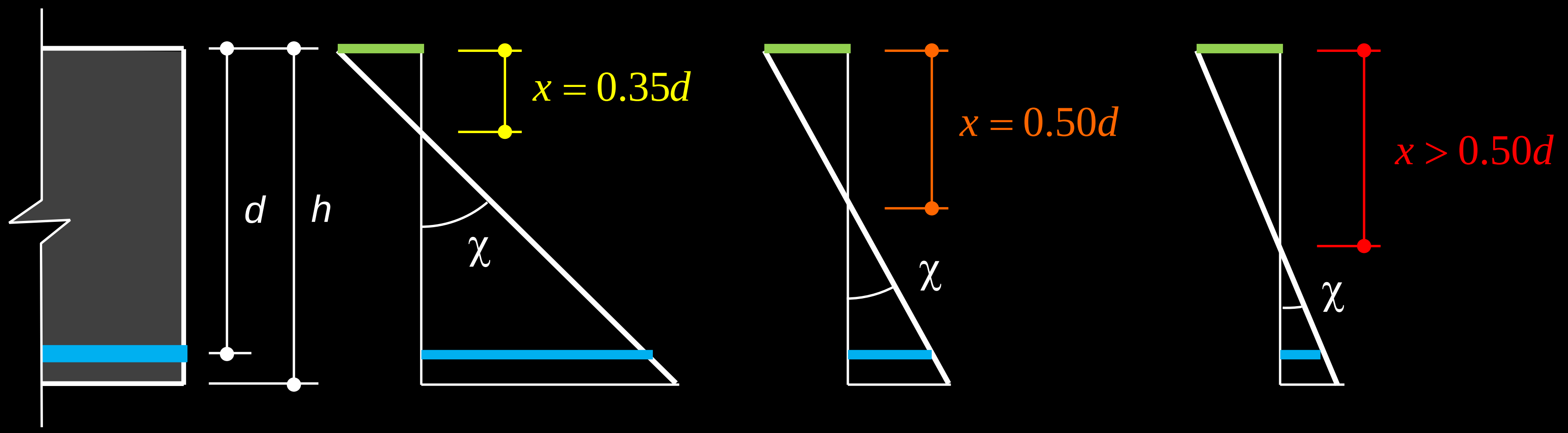


Maximum reinforcement ratio and bending resistance according to SIA 262, section 4.1.4.2:
(for components mainly subjected to bending)

- **x/d > 0.50: is to be avoided**

Beams – Deformation capacity

Limitation of the compression zone depth according to SIA 262



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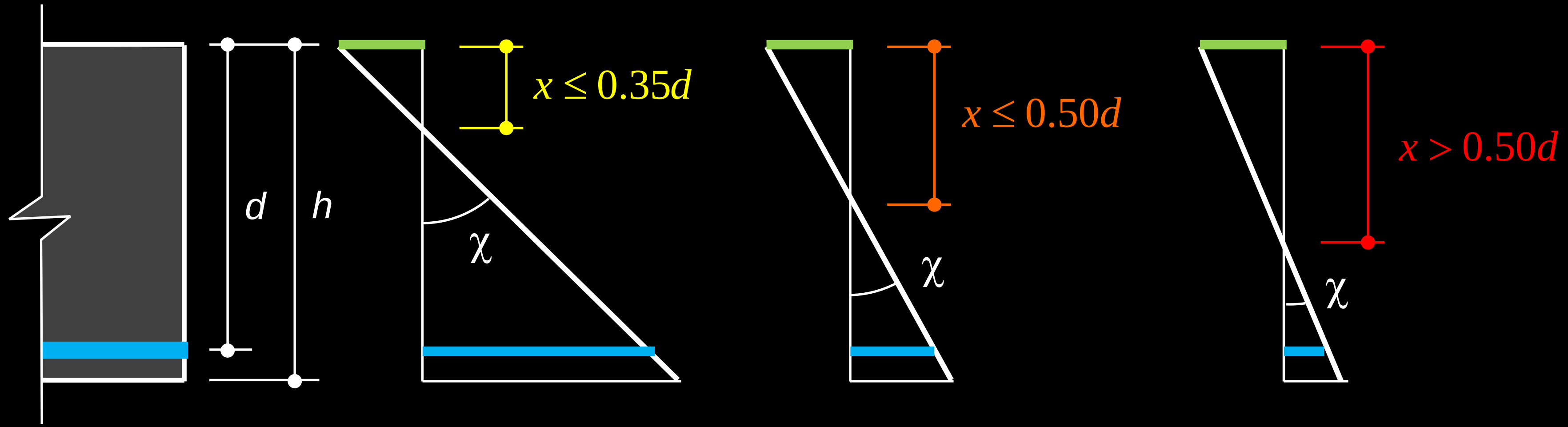
- 0.35 ≤ x/d ≤ 0.5: Internal force redistributions **with** verification of deformation capacity

  $$x/d \leq 0.50 \rightarrow \omega \leq 0.425 \rightarrow M_{Rd} \leq bd^2 f_{cd} \omega \cdot (1-\omega/2) = 0.335 \cdot bd^2 f_{cd}$$

- **x/d > 0.50: is to be avoided**

Beams – Deformation capacity

Limitation of the compression zone depth according to SIA 262



Maximum reinforcement ratio and bending resistance according to SIA 262, section 4.1.4.2:
(for components mainly subjected to bending)

- 0.35 ≤ x/d ≤ 0.5: Internal force redistributions **with** verification of deformation capacity

Beams – Deformation capacity

System behaviour
(see also [6], page 2.32ff)

Continuous increase of the load *q*:

→ Yielding begins first at the fixed-end support, forming a plastic hinge

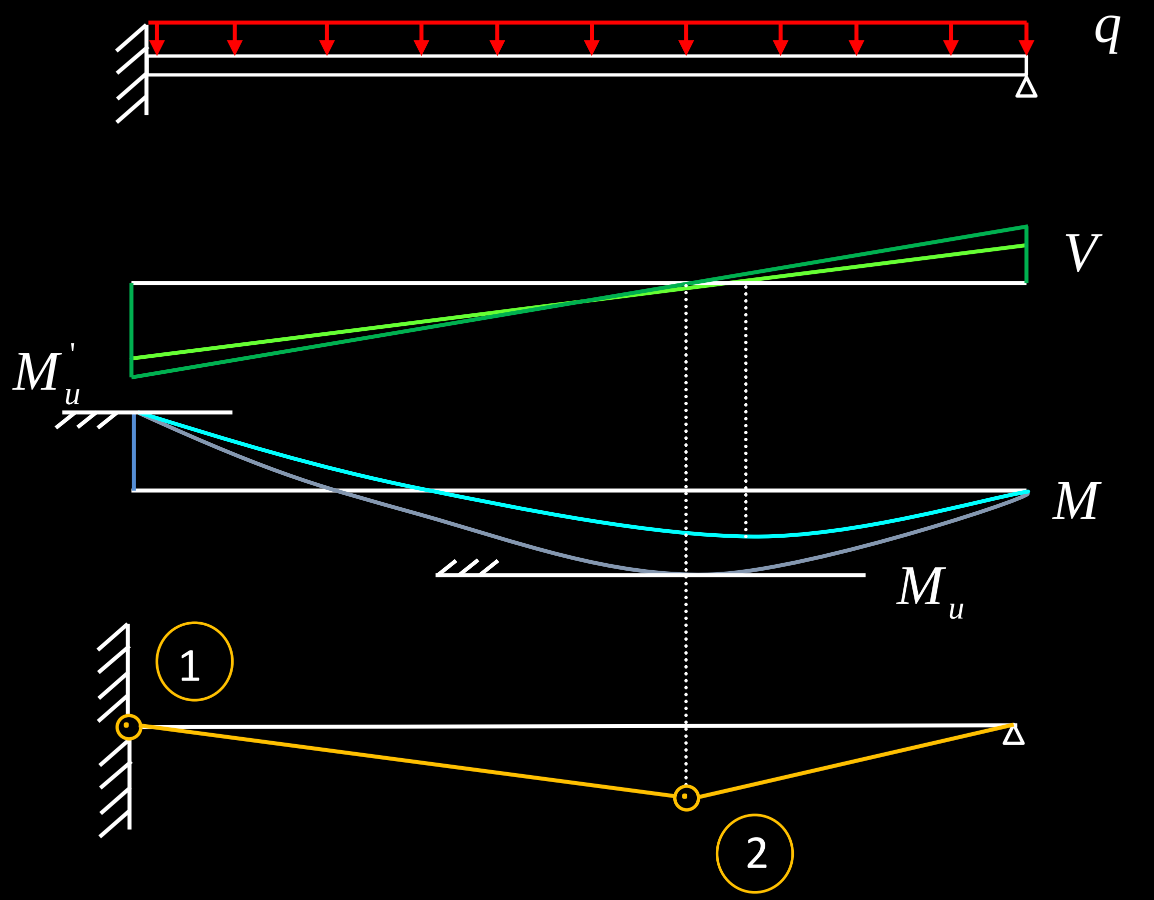
→ The statically indeterminate system turns (for additional loading) into a simple beam

Further load increase is possible until a second plastic hinge is formed in the member (= mechanism):

→ Plastic rotation required at the fixed support

→ **Rotation demand** depending on static system and load configuration

→ **Rotation capacity** limited by steel elongation and / or concrete compression



Verification = Comparison:
Deformation capacity Θ_{pu} ↔ Deformation demand $\Theta_{pu,dem}$

Beams – Deformation capacity

Rotation demand $\Theta_{pu,dem}$ (approximation, example two-span beam)

In general, deformation capacity and deformation demand are coupled.

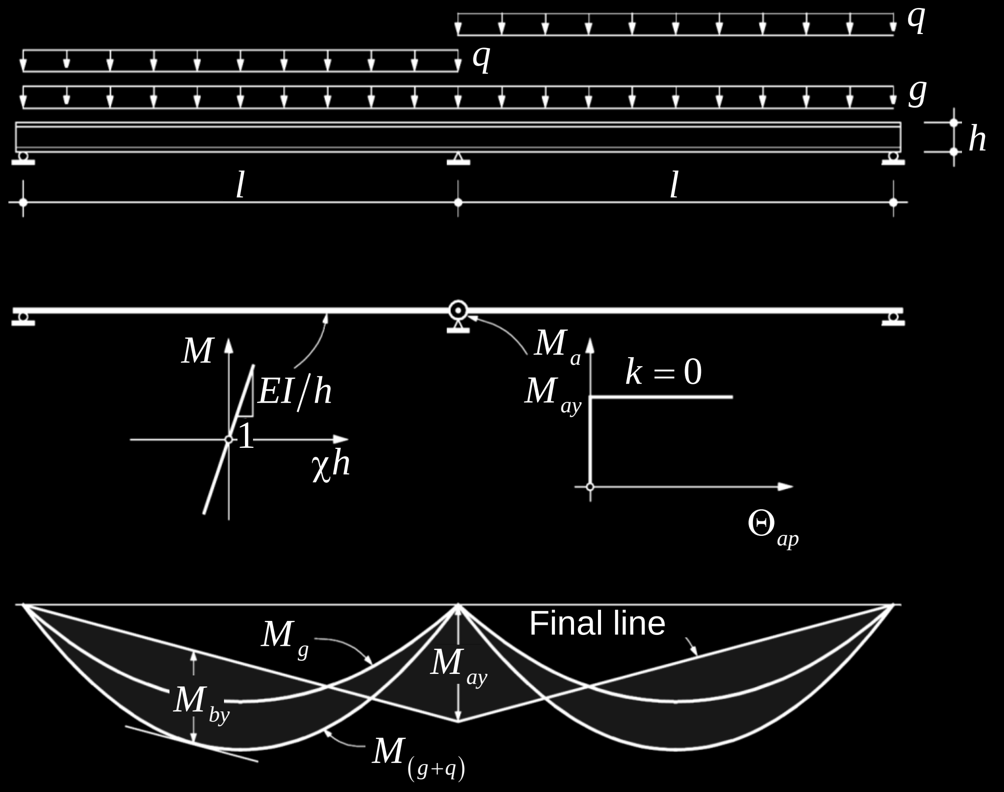
The interaction can only be neglected for moderate redistributions.

Additional simplifications:

- Constant bending stiffness
- M-Θ rigid-ideal plastic (no hardening in the plastic hinge)

Therefore, the rotation demand $\Theta_{pu,dem}$ of the intermediate support corresponds to the relative rotation of the two beams over the intermediate support, which can be considered as simply supported beams after reaching M_{ay}
(at $q = q_y$):

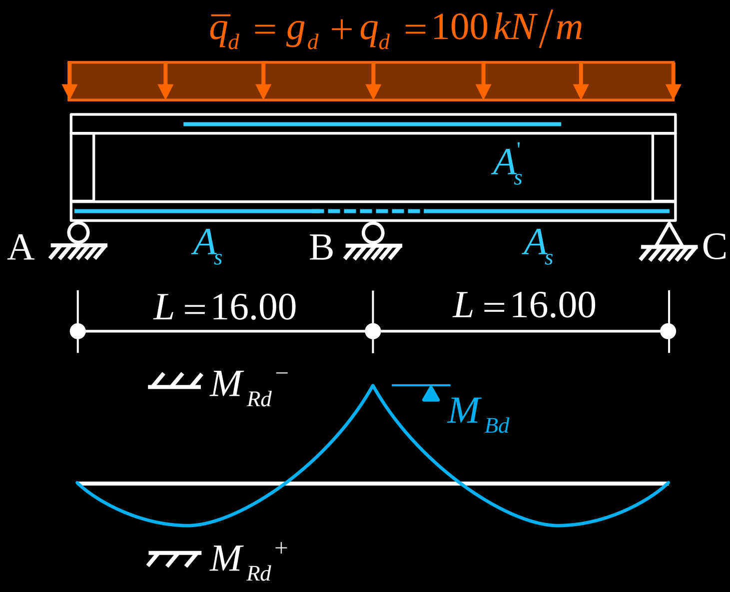
$$\Theta_{pu,dem} = \frac{(q - q_y)\, l^3}{12EI}$$

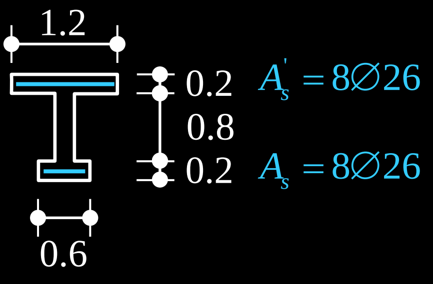


(Two-span beam, first plastic hinge at intermediate support, deformation demand for full load)

Beams – Deformation capacity

Rotation demand - Example of a two-span beam

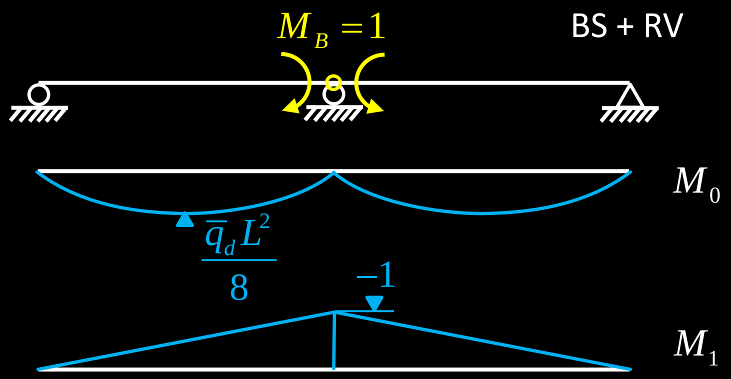




$A_s f_{sd} = 8 \cdot 530 \cdot 0.435 = 1848$ kN

$M_{Rd}^{\pm} = z \cdot A_s f_{sd} \approx 1848$ kNm

Moment at intermediate support



Force method

$$\Theta_{B0} = \int \frac{M_0 M_1}{EI} = 2 \cdot \frac{\bar{q}_d L^2}{8} \cdot \left(-\frac{L}{3EI^+}\right) = -\frac{\bar{q}_d L^3}{12EI^+}$$

$$\Theta_{B1} = \int \frac{M_1^2}{EI} = 2 \cdot (-1) \cdot (-1) \cdot \frac{L}{3EI^-} = \frac{2L}{3EI^-}$$

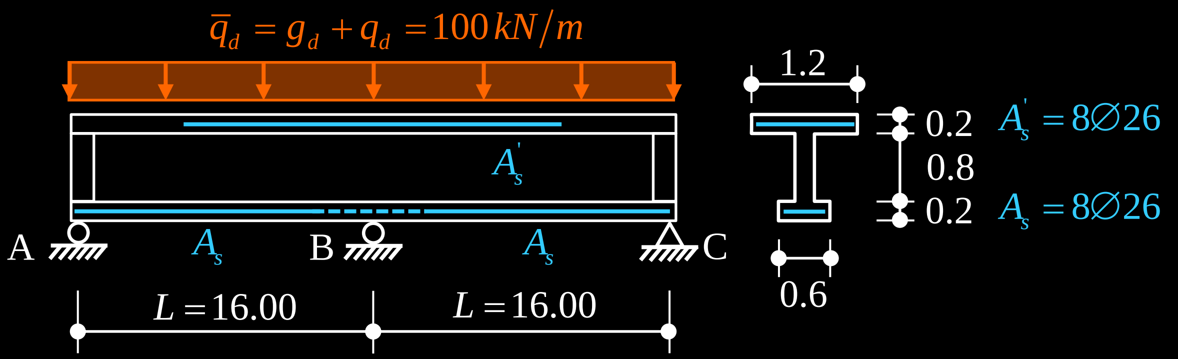
$$\Theta_B = \Theta_{B0} + M_B \Theta_{B1} = 0$$

$$\rightarrow M_B = -\frac{\Theta_{B0}}{\Theta_{B1}} = \frac{\bar{q}_d L^2}{8} \frac{EI^-}{EI^+} = \alpha_r \frac{\bar{q}_d L^2}{8} \leq \frac{\bar{q}_d L^2}{8} \text{(i.d.R.)}$$

Since $EI^- < EI^+$ (crack formation starts above B), part of the redistribution of internal forces starts before yielding (this reduces the plastic rotation demand $\rightarrow$ favourable).

Beams – Deformation capacity

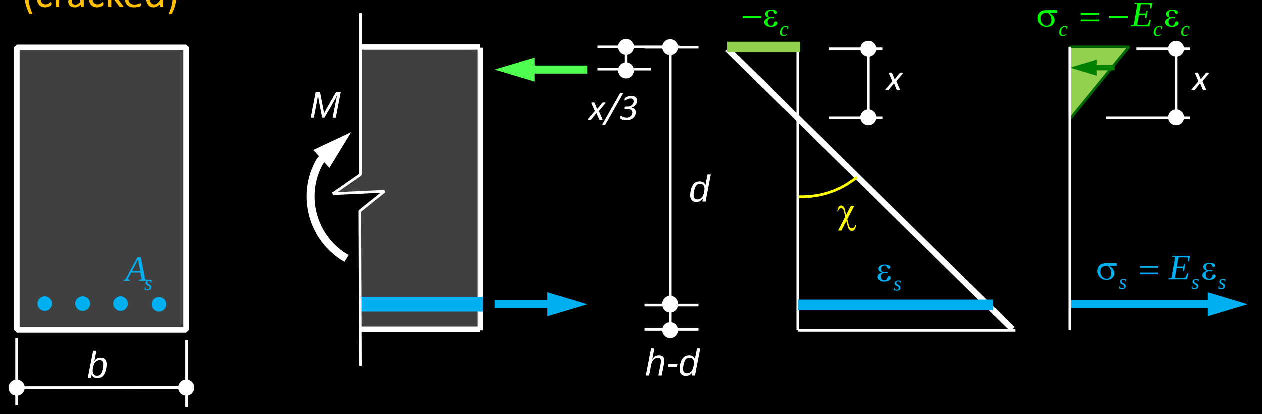
Rotation demand - Example of a two-span beam



$A_s f_{sd} = 8 \cdot 530 \cdot 0.435 = 1848$ kN

$M_{Rd}^{\pm} = z \cdot A_s f_{sd} \approx 1848$ kNm

EI^{II} (cracked)



$M = A_s E_s \varepsilon_s \left(d - \frac{x}{3}\right),\ \chi = \frac{\varepsilon_s}{d-x} = \frac{M}{EI^{II}}$ (here for simplicity $\varepsilon_{sm} = \varepsilon_{sr}$ is assumed, with $\varepsilon_{sm} < \varepsilon_{sr}$ a smaller rotation demand results)

$$\rightarrow EI^{II} = \frac{M}{\chi} = A_s E_s \underbrace{(d - x/3)}_{\approx z} \underbrace{(d - x)}_{\approx 0.9z} \approx 0.9 A_s E_s z^2 = 0.9 \cdot 4240 \cdot 205'000 \cdot 1^2 = 780\ \text{MNm}^2\ \ (EI_i^I = 3502\ \text{MNm}^2)$$

Beams – Deformation capacity

Rotation demand - Example of a two-span beam

Yielding

$$\alpha_r \frac{\bar{q}_d L^2}{8} = M_{Rd}^- \quad \rightarrow \bar{q}_{dy} = \frac{8M_{Rd}^-}{\alpha_r L^2} = \frac{1}{\alpha_r}\frac{8\cdot 1848}{256}$$

$$= \frac{1}{\alpha_r} 57.8 \text{ kNm}^{-1}$$

$$\rightarrow \bar{q}_d - \bar{q}_{dy} = 100 - \frac{1}{\alpha_r} 57.8 kNm^{-1} = 42.2 \text{ kNm}^{-1} \; (\alpha_r = 1.0)$$

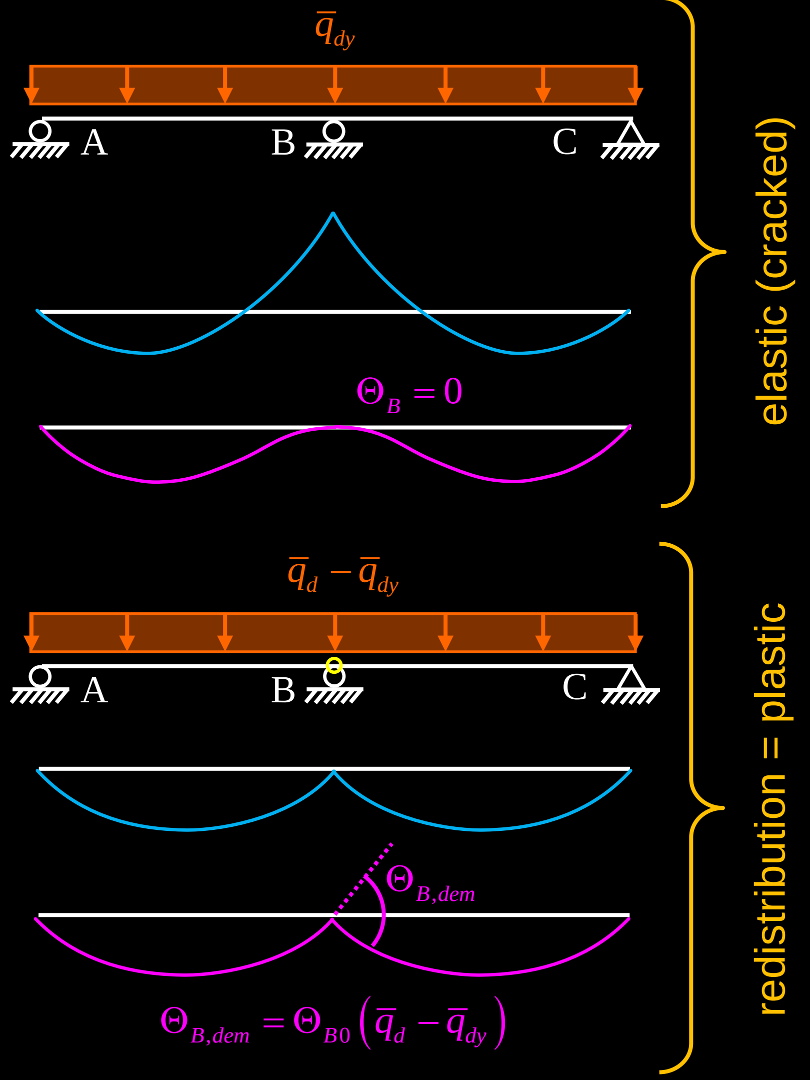
$$= 27.8 \text{ kNm}^{-1} \; (\alpha_r = 0.8)$$

$$\Theta_{B,dem} = \left(\bar{q}_d - \bar{q}_{dy}\right)\frac{L^3}{12EI} = \frac{42.2\cdot 16^3}{12\cdot 780\cdot 10^3}\frac{\text{kNm}^2}{\text{kNm}^2}$$

$$= 18.5 \text{ mrad} \quad (\alpha_r = 1)$$

$$= 12.2 \text{ mrad} \quad (\alpha_r = 0.8)$$

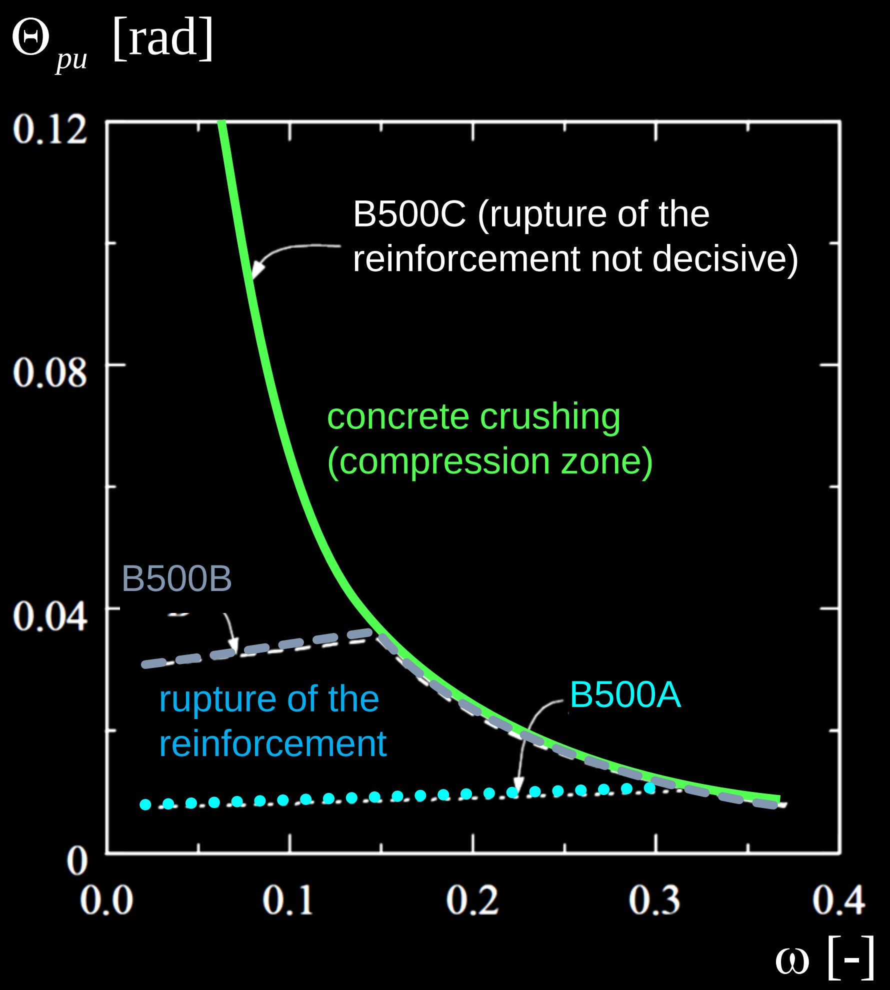
After reaching M_{Rd}^-:
Two simply supported beams for additional loading $\bar{q}_d - \bar{q}_{dy}$ with the corresponding relative rotation of the beam ends at B (see BS+RV in slide 33)

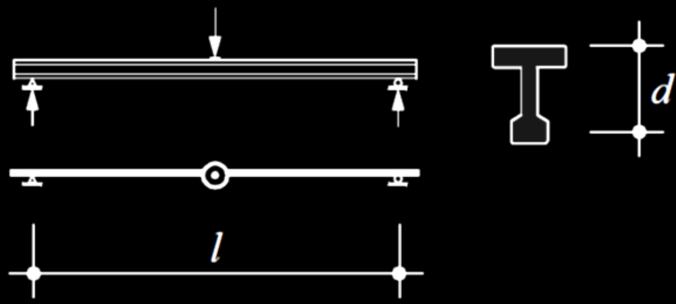


Beams – Deformation capacity

Rotation capacity Θ_{pu} – Basics

Example: Plastic hinge angle as a function of ω (ductility classes A-C, 1999)





Basis of the calculations:

f_y = 500 MPa	l/d = 20
E_s = 200 GPa	θ = 45 °
f_c = 30 MPa	$\varnothing$ = 20 mm
ε_{cu} = 5 ‰	s_{rm} = 150 mm

Beams – Deformation capacity

Rotation capacity Θ_{pu} (simplified) (see also [6], page 2.32ff)

Limitation of the plastic rotation by the reinforcing steel (rupture of the reinforcement):

$$\Theta_{pus} = L_{pl} \cdot \left(\frac{\varepsilon_{smu}}{d-x} - \frac{\varepsilon_{smy}}{d-x} \right)$$

Curvature at onset of yielding

Curvature at rupture of the reinforcement

Limitation of the plastic rotation by the concrete (compressive failure):

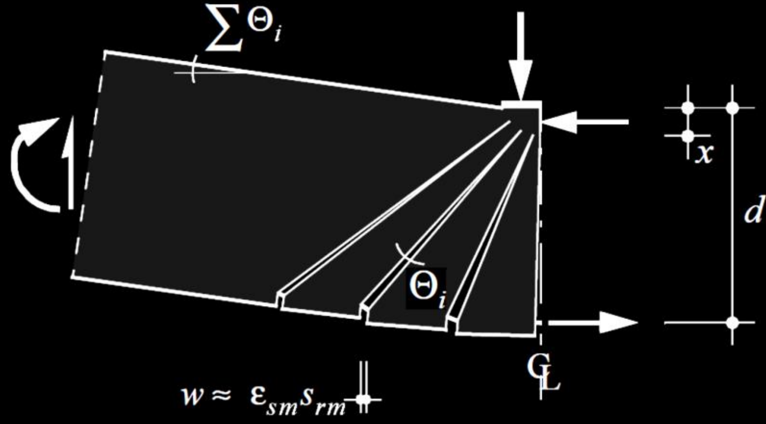
$$\Theta_{puc} = L_{pl} \left(\frac{\varepsilon_{c2d}}{x} - \frac{\varepsilon_{smy}}{d-x} \right)$$

Curvature at onset of yielding

Curvature at concrete crushing

Rotation per crack: $\Theta_i \approx \dfrac{\varepsilon_{sm} s_{rm}}{d-x}$

Plastic hinge rotation = sum of the plastic rotations of all cracks from the onset of yielding



L_{pl} Plastic hinge length, depending on load configuration and geometry: region in which the chord reinforcement yields ($\rightarrow$ determine the chord force distribution from the stress field).

ε_{smu} Mean steel elongation when reaching $\varepsilon_{sr} = \varepsilon_{ud}$ $\sigma_{sr} = f_t$

ε_{smy} Mean steel elongation when reaching $\varepsilon_{sr} = \dfrac{f_s}{E_s}$ $\sigma_{sr} = f_s$

$\varepsilon_{sr} \leftrightarrow \varepsilon_{sm}$ tension chord model (Stahlbeton I)

Beams – Deformation capacity

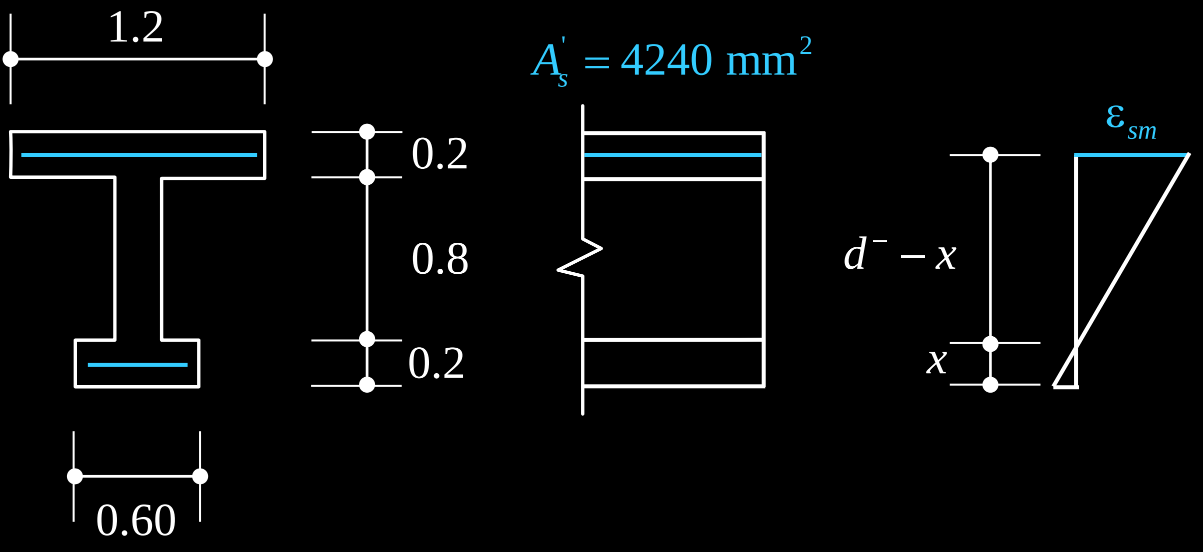
Rotation demand ↔ Rotation capacity (simplified) - Example of a two-span beam

- C30/37:
  f_{cd} = 20 MPa, f_{ctm} = 2.9 MPa
- $d^{-} \approx 1.1\ \text{m},\ A_s' f_{sd} = 1848\ \text{kN}$

  $\rightarrow x = \dfrac{1848}{0.85 \cdot 0.6 \cdot 20} = 181\ \text{mm}$

  $\dfrac{x}{d} = 0.16$ → Verification not required (see notes *)

  $d^{-} - x = 919\ \text{mm}$



Rotation at failure:

$$\left.\begin{aligned}\Theta_{puc} &= L_{pl} \cdot \left(\frac{\varepsilon_{cu}}{x} - \frac{\varepsilon_{smy}}{d^{-} - x} \right) \\ \Theta_{pus} &= L_{pl} \cdot \left(\frac{\varepsilon_{smu}}{d^{-} - x} - \frac{\varepsilon_{smy}}{d^{-} - x} \right)\end{aligned}\right\}$$

with $\dfrac{\varepsilon_{smy}}{d^{-} - x}$ = Curvature at onset of yielding $= \dfrac{f_s/E_s - \Delta\varepsilon_0}{d^{-} - x} = 2.3\,\text{mrad/m}$, L_{pl} = length plastic hinge $\approx 2d^{-}$

Beams – Deformation capacity

Rotation demand ↔ Rotation capacity (simplified) - Example of a two-span beam

Rotation at failure:

Concrete crushing

$$\Theta_{puc} = L_{pl} \cdot \left(\frac{\varepsilon_{cu}}{x} - \frac{\varepsilon_{smy}}{d^- - x} \right) \approx 2 \cdot 1.10 \cdot \left(\frac{0.003}{0.181} - 0.0023 \right) = 14.3 \; \frac{\text{mrad}}{\text{m}} \cdot 2.2 \text{ m} = 31.4 \text{ mrad}$$

$$\rightarrow \Theta_{puc} > \Theta_{B,dem} \rightarrow \text{OK}$$

Steel rupture

rough assumption: $\varepsilon_{smu} \approx 0.5\varepsilon_{ud} = \begin{cases} 22.5‰ \text{ (B500B)} \\ 32.5‰ \text{ (B500C)} \end{cases}$

(estimated reduction of elongation at failure due to tension stiffening - see next slides)

$$\Theta_{pus} = L_{pl} \cdot \left(\frac{\varepsilon_{smu}}{d^- - x} - \frac{\varepsilon_{smy}}{d^- - x} \right) = \begin{cases} 2 \cdot 1.10 \cdot \left(\frac{0.0225}{0.919} - 0.0023 \right) = 22.2 \frac{\text{mrad}}{\text{m}} \cdot 2.2 \text{ m} = 48.8 \text{ mrad (B500B)} \\ 2 \cdot 1.10 \cdot \left(\frac{0.0325}{0.919} - 0.0023 \right) = 33.1 \frac{\text{mrad}}{\text{m}} \cdot 2.2 \text{ m} = 72.7 \text{ mrad (B500C)} \end{cases}$$

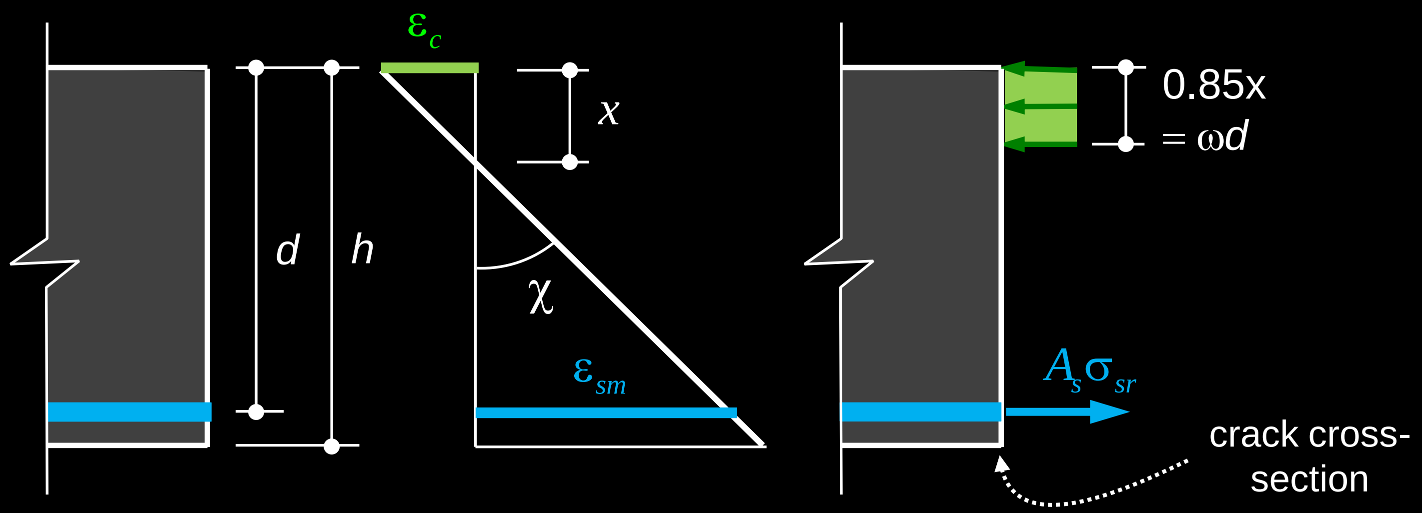
$$\rightarrow \Theta_{pus} > \Theta_{B,dem} \rightarrow \text{OK}$$

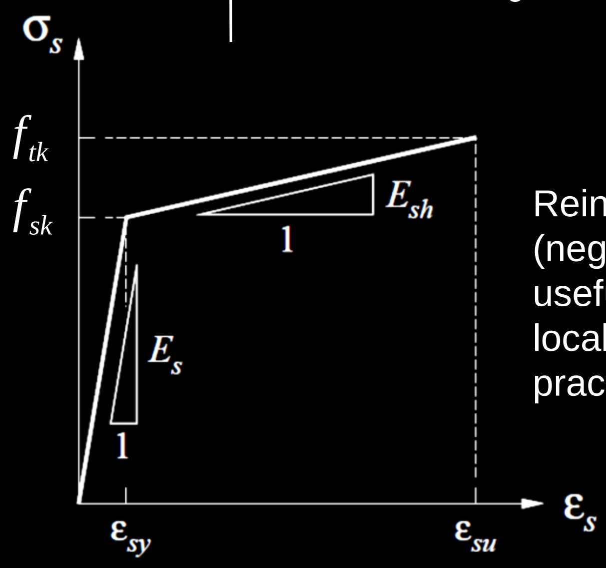
The rotation capacity would be verified.

But: Are the assumptions of L_{pl}, ε_{smu} all right?

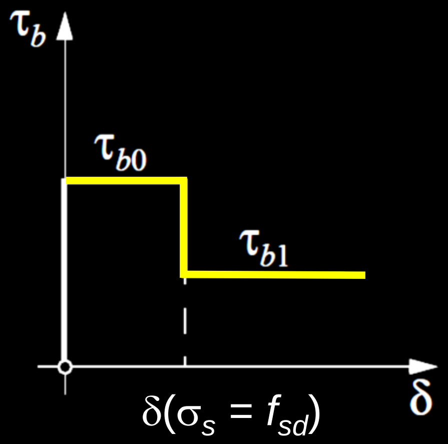
Beams – Deformation capacity

Rotation capacity Θ_{pu} (detailed investigation) - Basics





Reinforcing steel with hardening (neglecting hardening here is not useful: plastic deformations would localise in a single crack, leading to practically no rotation capacity)



Tension chord model (already presented)

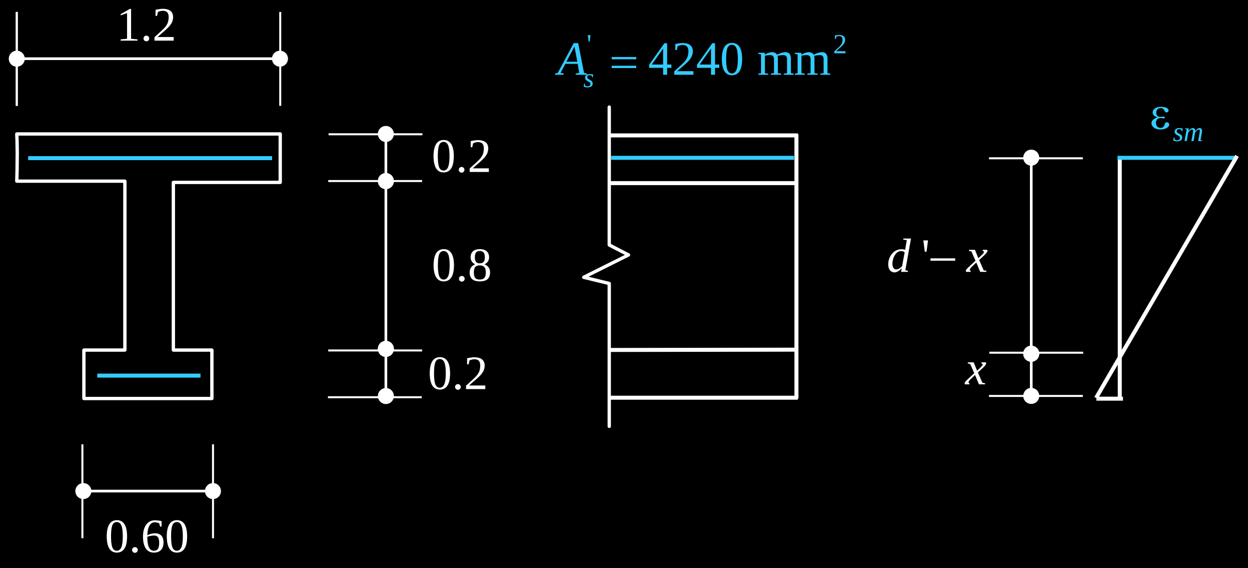
Beams – Deformation capacity

Rotation capacity (detailed investigation) - Example of a two-span beam

- C30/37:
  $f_{cd} = 20\text{ MPa},\ f_{ctm} = 2.9\text{ MPa}$
- $d^- \approx 1.1\text{ m},\ A_s' f_{sd} = 1848\text{ kN}$

$$\rightarrow x = \frac{1848}{0.85 \cdot 0.6 \cdot 20} = 181\text{ mm}$$

$$d^- - x = 919\text{ mm}$$



Equivalent reinforcement ratio
(considering *x* at failure, see notes *):

$$\rho_{eff} = \frac{1}{\dfrac{M_r (d^- - x) E_s}{f_{ct} EI^{II}} + 1 - n} = 2.2\%$$

$$s_{rm0} \approx \frac{\varnothing}{4} \cdot \left(\frac{1}{\rho_t} - 1 \right) = 292\text{ mm} \quad \left(\lambda = \frac{1}{2} \ldots 1 \right)$$

$\rightarrow s_{rm} \approx 250\text{ mm}$ (spacing of stirrups)

Beams – Deformation capacity

Rotation capacity (detailed investigation) - Example of a two-span beam

Tension chord model

$\varnothing = 26$ mm

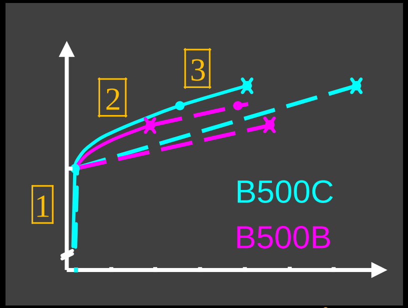
$s_{rm} = 250$ mm (spacing of stirrups)

$E_s = 205$ GPa

$f_{ctm} = 2.9$ MPa

$\tau_{b0} = 2 f_{ctm} = 5.8$ MPa

$\tau_{b1} = 1 f_{ctm} = 2.9$ MPa



[1] $\sigma_{sr} < f_s$ "elastic"

$$\rightarrow \varepsilon_{sm} = \underbrace{\frac{\sigma_{sr}}{E_s}}_{\text{nackter Stahl}} - \underbrace{\frac{\tau_{b0} s_r}{E_s \varnothing}}_{\Delta\varepsilon_0 = \Delta\varepsilon^{[1]}} = \frac{\sigma_{sr}}{E_s} - 0.27‰$$

[2] $\sigma_{sr} > f_s$, $\sigma_{smin} < f_s$ $\rightarrow$ ...; Transition to regime [3] at

"partially yielded"

$$\sigma_{smin} = \sigma_{sr} - \frac{2\tau_{b1} s_r}{\varnothing} = \sigma_{sr} - 56 \text{ MPa} \overset{!}{=} f_s$$

$\rightarrow \sigma_{sr} = f_s + 56$ MPa $\rightarrow$ B500B stays at regime [2]

[3] $\sigma_{smin} > f_s$

"fully yielded"

$$\rightarrow \varepsilon_{sm} = \underbrace{\frac{f_s}{E_s} + \frac{\sigma_{sr} - f_s}{E_{sh}}}_{\text{nackter Stahl}} - \underbrace{\frac{\tau_{b1} s_r}{E_{sh}\varnothing}}_{\Delta\varepsilon_1 = \Delta\varepsilon^{[3]}}$$

23‰ (B500C)
30‰ (B500B)

$$\sigma_{sr} = f_{sd} + E_{sh}\left(\varepsilon_m - \frac{f_s}{E_s}\right) + \underbrace{\frac{\tau_{b1} s_r}{\varnothing}}_{29 MPa}$$

B500C: $\varepsilon_{sm}(\sigma_{sr} = f_s) = 2.43 - 0.27 = 2.16‰$

$\varepsilon_{sm}(\sigma_{smin} = f_s) = 25.9‰$ ([3] with $\sigma_{sr} = 556$ MPa)

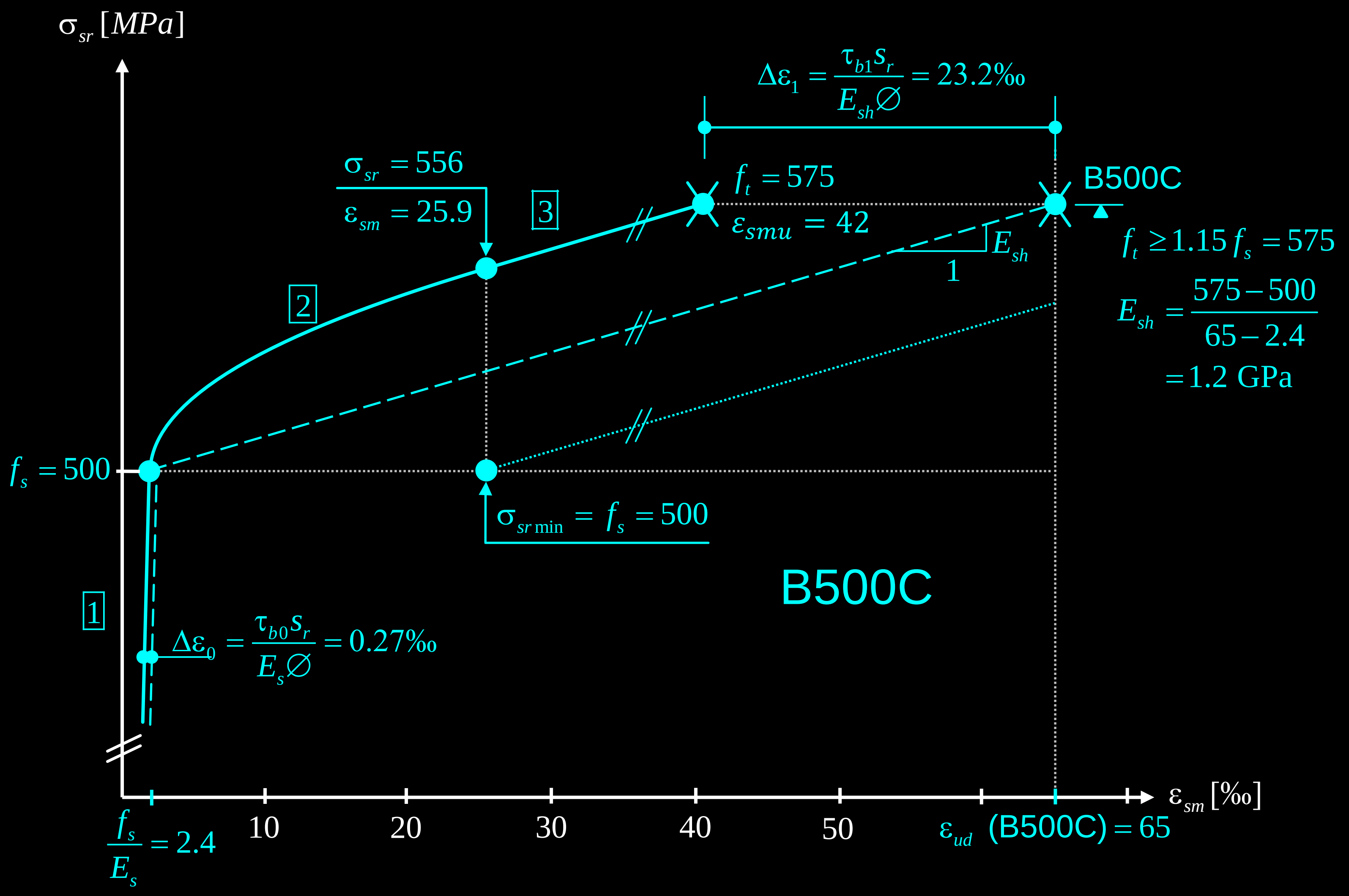
$\varepsilon_{sm}(\sigma_{sr} = f_t) = 65 - 23 = 42‰ = \varepsilon_{smu}$

B500B: $\varepsilon_{sm}(\sigma_{sr} = f_s) = 2.16‰$

$\varepsilon_{smu} = 17.7‰$ (Regime [2] with $\sigma_{sr} = f_t$, does not reach regime [3])

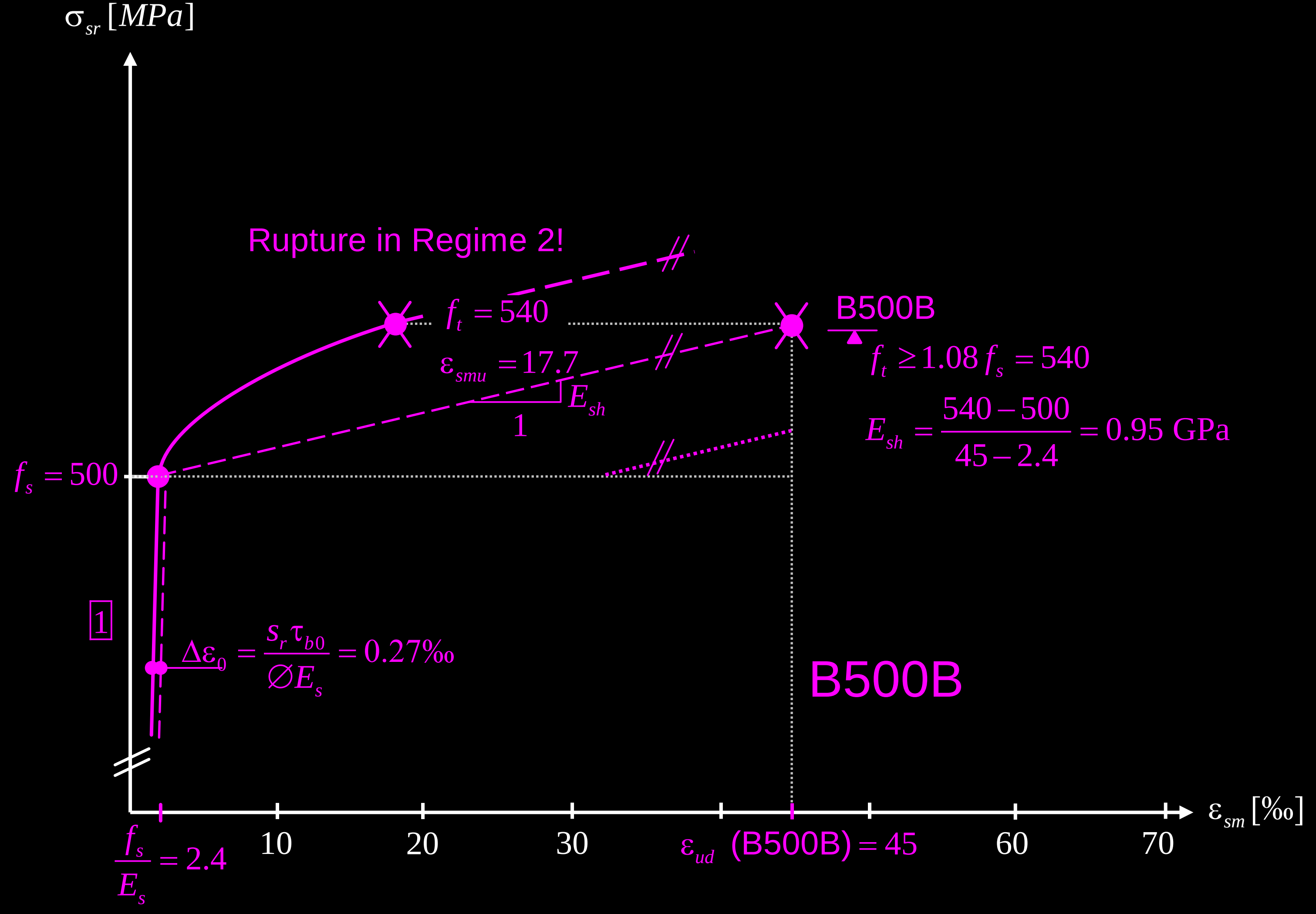
Beams – Deformation capacity

Rotation capacity (detailed investigation) - Example of a two-span beam



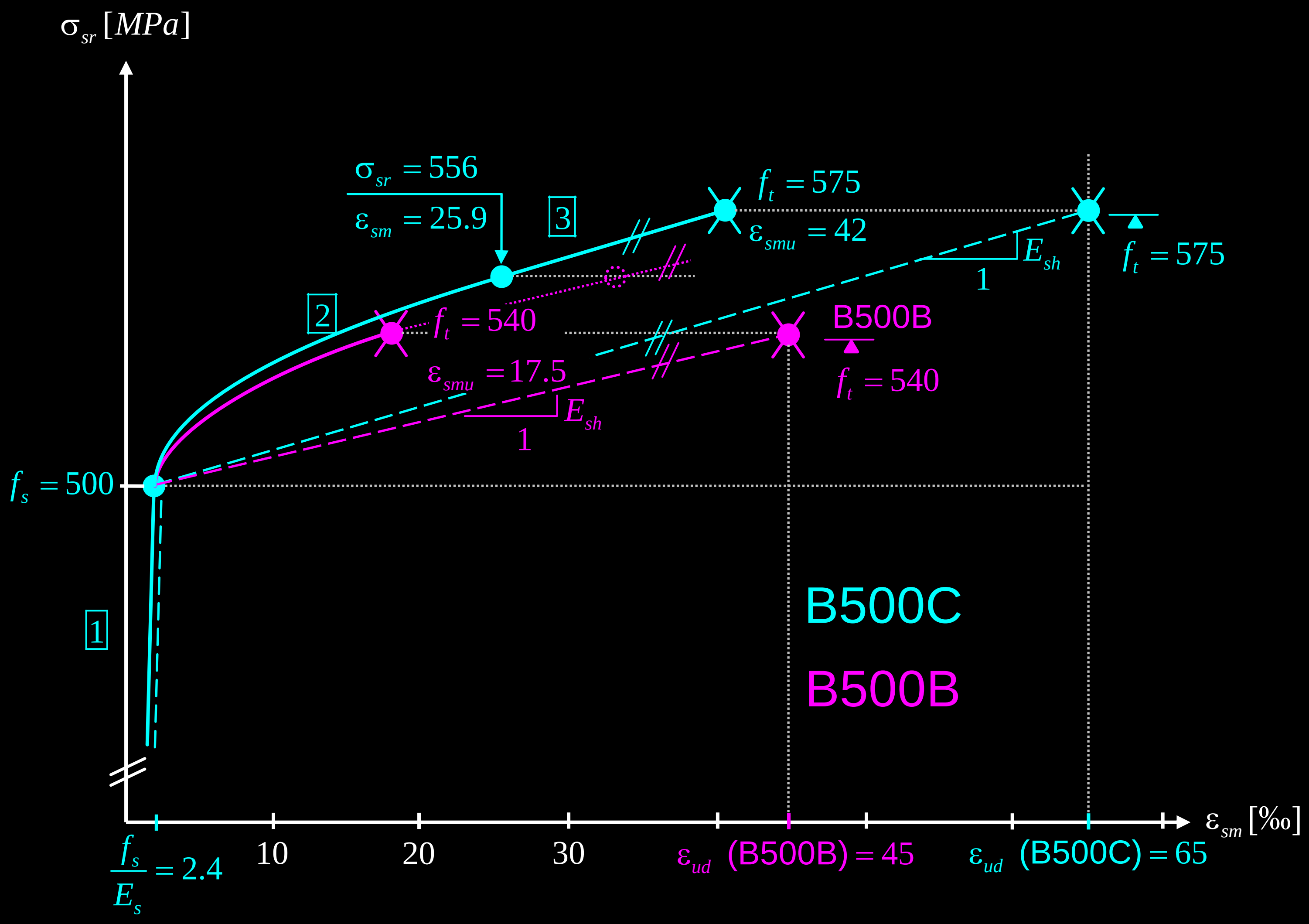
Beams – Deformation capacity

Rotation capacity (detailed investigation) - Example of a two-span beam



Beams – Deformation capacity

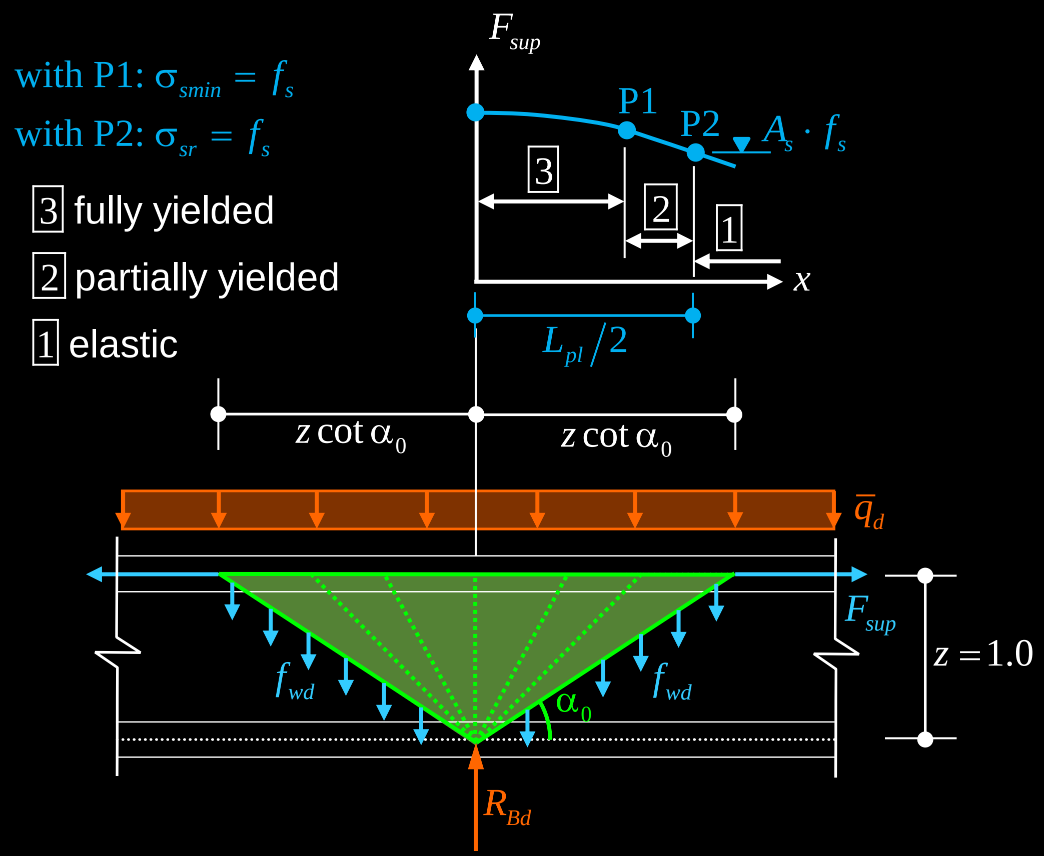
Rotation capacity (detailed investigation) - Example of a two-span beam



Beams – Deformation capacity

Rotation capacity (detailed investigation) - Example of a two-span beam

Plastic hinge length → Distribution of the top chord force F_{sup} determined from a stress field



$$\frac{dF_{sup}}{dx} = -\left(\bar{q}_d + f_{wd}\right)\cot\alpha_0(x)$$

$$\cot\alpha_0(x) = x/z$$

$$F_{sup}(x) = A_s f_t - \frac{x^2}{2}\frac{\left(\bar{q}_d + f_{wd}\right)}{z}$$

$$\sigma_{sr} = f_s: \quad F_{sup} = A_s f_s \overset{!}{=} A_s f_t - \frac{x^2}{2}\frac{\left(\bar{q}_d + f_{wd}\right)}{z}$$

$$\rightarrow x_{P2} = \sqrt{\frac{2A_s\left(f_t - f_s\right)z}{\bar{q}_d + f_{wd}}}$$

$$\sigma_{smin} = f_s, \; \sigma_{sr} = f_s + \frac{2\tau_{b1}s_{rm}}{\varnothing}$$

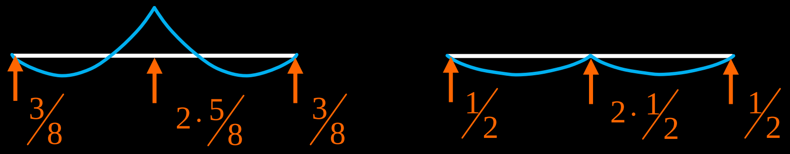
$$\rightarrow x_{P1} = \sqrt{\frac{2A_s\left(f_t - f_s - \frac{2\tau_{b1}s_{rm}}{\varnothing}\right)z}{\bar{q}_d + f_{wd}}}$$

Beams – Deformation capacity

Rotation capacity (detailed investigation) - Example of a two-span beam

Plastic hinge length → Distribution of the top chord force F_{sup} determined from a stress field

R_{Bd} increases during the redistribution, x_P thus decreases (large gradient of M is unfavourable for the rotational capacity, since a stronger localization of deformations occurs):



$$R_{Bd} = 2 \cdot \frac{5}{8} \cdot L \cdot \bar{q}_{dy} + 2 \cdot \frac{1}{2} \cdot L \cdot \left(\bar{q}_d - \bar{q}_{dy}\right)$$

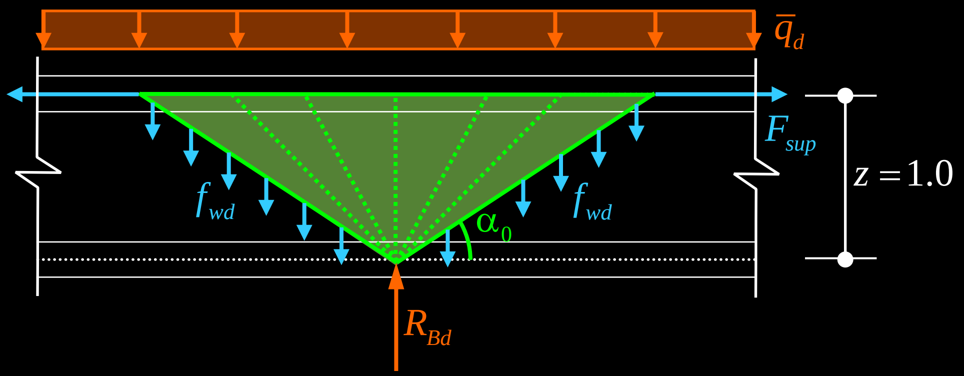
$$= \underbrace{1156\ \text{kN}}_{\text{Reaction for } \bar{q}_{dy} = \text{Begin of redistribution}} + \underbrace{675\ \text{kN}}_{\text{additional for } \bar{q}_d > \bar{q}_{dy}} = 1831\ \text{kN}$$

(für $EI^- = EI^+$)

R_{Bd} (and thus x_P) also depends on the choice of the compression field inclination α_0:

$$R_{Bd} = 2z\cot(\alpha_0)(\bar{q}_d + f_{wd}) \rightarrow (\bar{q}_d + f_{wd}) = \frac{R_{Bd}}{2z\cot(\alpha_0)}$$

large $\alpha_0 \rightarrow x_{P1}, x_{P2}$ small, small $\alpha_0 \rightarrow x_{P1}, x_{P2}$ large



→ Several assumptions are necessary to determine the deformation capacity

→ Rough estimation, not exact calculation!

Beams – Deformation capacity

Rotation capacity (detailed investigation) - Example of a two-span beam

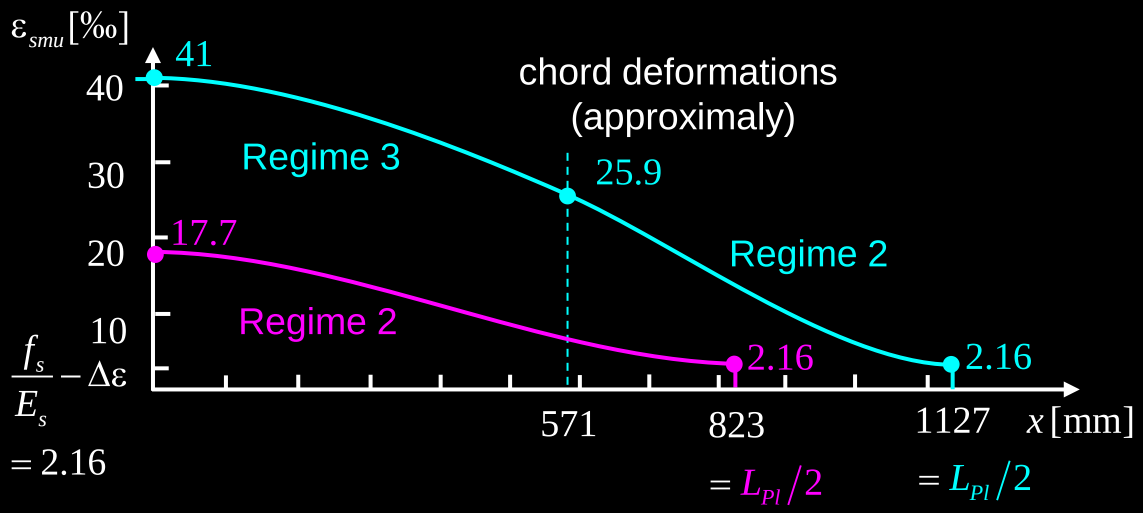
Plastic hinge length → Distribution of the top chord force F_{sup} determined from a stress field

→ Assumption: $R_{Bd} \approx 1500\ \text{kN},\ \cot(\alpha_0) = 1.5\ (\alpha_0 = 33.5°),\ (\bar{q}_d + f_{wd}) = \frac{1500}{2\cot(\alpha_0)} = \frac{1500}{3} = 500\ \text{kNm}^{-1}$

B500B: $x_{P2} = \sqrt{\frac{2 \cdot 4240(540-500) \cdot 1000}{500}} = 823\ \text{mm}\ ("L_{Pl}/2")$

B500C: $x_{P2} = \sqrt{\frac{2 \cdot 4240(575-500) \cdot 1000}{500}} = 1127\ \text{mm}\ ("L_{Pl}/2")$

$x_{P1} = \sqrt{\frac{2 \cdot 4240(575-556) \cdot 1000}{500}} = 571\ \text{mm}$



B500B: $\bar{\varepsilon}_{smu} = \frac{2}{L_{pl}} \cdot \int_0^{x_{p2}} \varepsilon_{sm}(x) \cdot dx$

$= 10.5‰$ (averaged over $L_{pl} = 1.65$ m)

B500C: $\bar{\varepsilon}_{smu} \approx \frac{2}{L_{pl}} \cdot \left(\int_0^{x_{p1}} \varepsilon_{sm}(x) \cdot dx + \int_{x_{p1}}^{x_{p2}} \varepsilon_{sm}(x) \cdot dx \right)$

$= 24.1‰$ (averaged over $L_{pl} = 2.25$ m)

Beams – Deformation capacity

Rotation demand and rotation capacity (detailed investigation) - Example two-span beam

Plastic rotation at failure

Concrete crushing

$$\Theta_{puc} = L_{pl} \cdot \left(\frac{\varepsilon_{cu}}{x} - \frac{\varepsilon_{smy}}{d^{-} - x} \right) \approx 2 \cdot 1.10 \cdot \left(\frac{0.003}{0.181} - 0.0023 \right) = 14.3 \ \frac{\text{mrad}}{\text{m}} \cdot 2.2 \text{ m} = 31.4 \text{ mrad}$$

$$\rightarrow \Theta_{puc} > \Theta_{B,req} \rightarrow \text{OK}$$

Steel rupture

Rough assumption:

$$\varepsilon_{smu} \approx 0.5\varepsilon_{ud} = \begin{cases} 22.5‰ \text{ with } L_{pl} = 2.2 \text{ m (B500B)} \\ 32.5‰ \text{ with } L_{pl} = 2.2 \text{ m (B500C)} \end{cases}$$

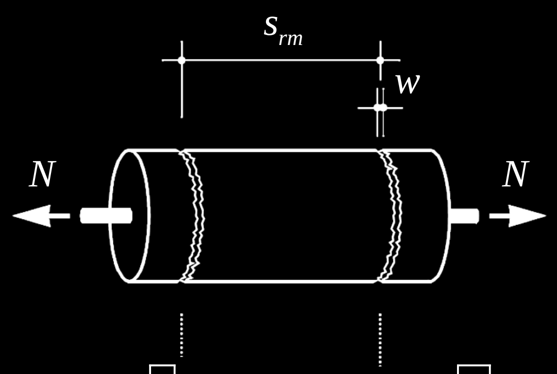
More detailed investigation:

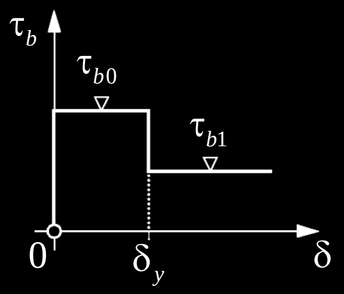
$$\bar{\varepsilon}_{smu} = \begin{cases} 10.5‰ \text{ with } L_{pl} = 1.65 \text{ m (B500B)} \\ 24.1‰ \text{ with } L_{pl} = 2.25 \text{ m (B500C)} \end{cases} \quad \begin{cases} \bar{\varepsilon}_{smu} \approx 0.23 \cdot \varepsilon_{ud}, \ L_{pl} \approx 1.5 \cdot d \\ \bar{\varepsilon}_{smu} \approx 0.37 \cdot \varepsilon_{ud}, \ L_{pl} \approx 2.0 \cdot d \end{cases}$$

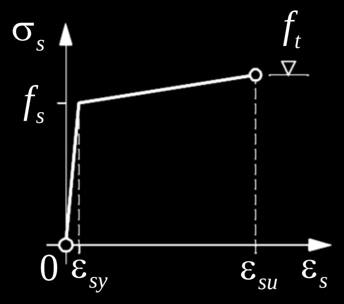
$$\Theta_{pus} = L_{pl} \cdot \left(\frac{\bar{\varepsilon}_{smu}}{d^{-} - x} - \frac{\varepsilon_{smy}}{d^{-} - x} \right) = \begin{cases} 1.65 \cdot \left(\frac{0.0105}{0.919} - 0.0023 \right) = 15.1 \text{ mrad (B500B)} < \Theta_{B,req} = 18.5 \text{ mrad } (\alpha_r = 1) \\ \text{not fulfilled!} \\ 2.25 \cdot \left(\frac{0.0241}{0.919} - 0.0023 \right) = 53.8 \text{ mrad (B500C)} \gg \Theta_{B,req} = 18.5 \text{ mrad } (\alpha_r = 1) \\ \text{ok (no problem)} \end{cases}$$

Beams – Deformation capacity

Additional considerations: ratio of mean strain to maximum strain in the cracks considering bond







1 Reinforcement elastic over entire length

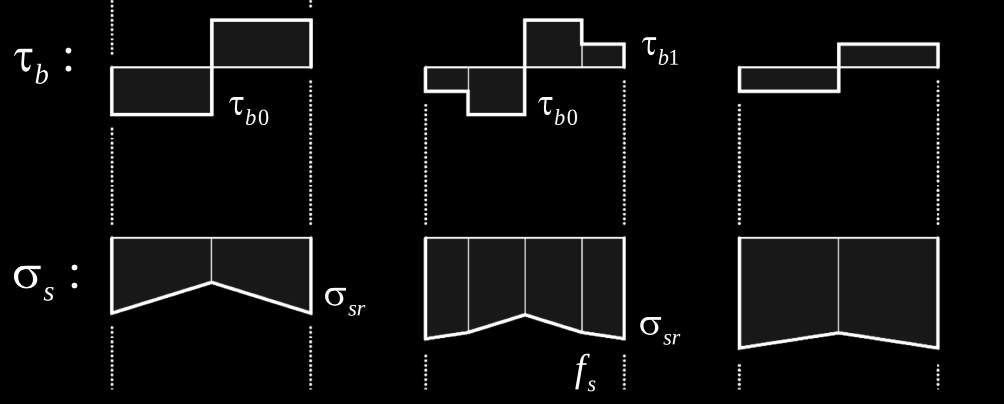
2 Reinforcement yielded near cracks

3 Reinforcement yielded over entire length

$\sigma_{sr} \le f_s$

$f_s \le \sigma_{sr} \le \left(f_s + \frac{2\tau_{b1} s_r}{\varnothing} \right)$

$\left(f_s + \frac{2\tau_{b1} s_r}{\varnothing} \right) \le \sigma_{sr} \le f_s$



Mean strains in the three regimes for bilinear characteristic curves of the reinforcement:

1 $$\varepsilon_{sm} = \frac{\sigma_{sr}}{E_s} - \frac{\tau_{b0} s_r}{E_s \varnothing}$$

2 $$\varepsilon_{sm} = \frac{(\sigma_{sr} - f_s)^2 \varnothing}{4 E_{sh} \tau_{b1} s_r} \cdot \left(1 - \frac{E_{sh} \tau_{b0}}{E_s \tau_{b1}} \right) + \frac{(\sigma_{sr} - f_s)}{E_s} \cdot \frac{\tau_{b0}}{\tau_{b1}} + \left(\varepsilon_{sy} - \frac{\tau_{b0} s_r}{E_s \varnothing} \right)$$

3 $$\varepsilon_{sm} = \frac{(\sigma_{sr} - f_s)}{E_{sh}} + \left(\varepsilon_{sy} - \frac{\tau_{b1} s_r}{E_{sh} \varnothing} \right)$$

[Alvarez 1999]

Beams – Deformation capacity

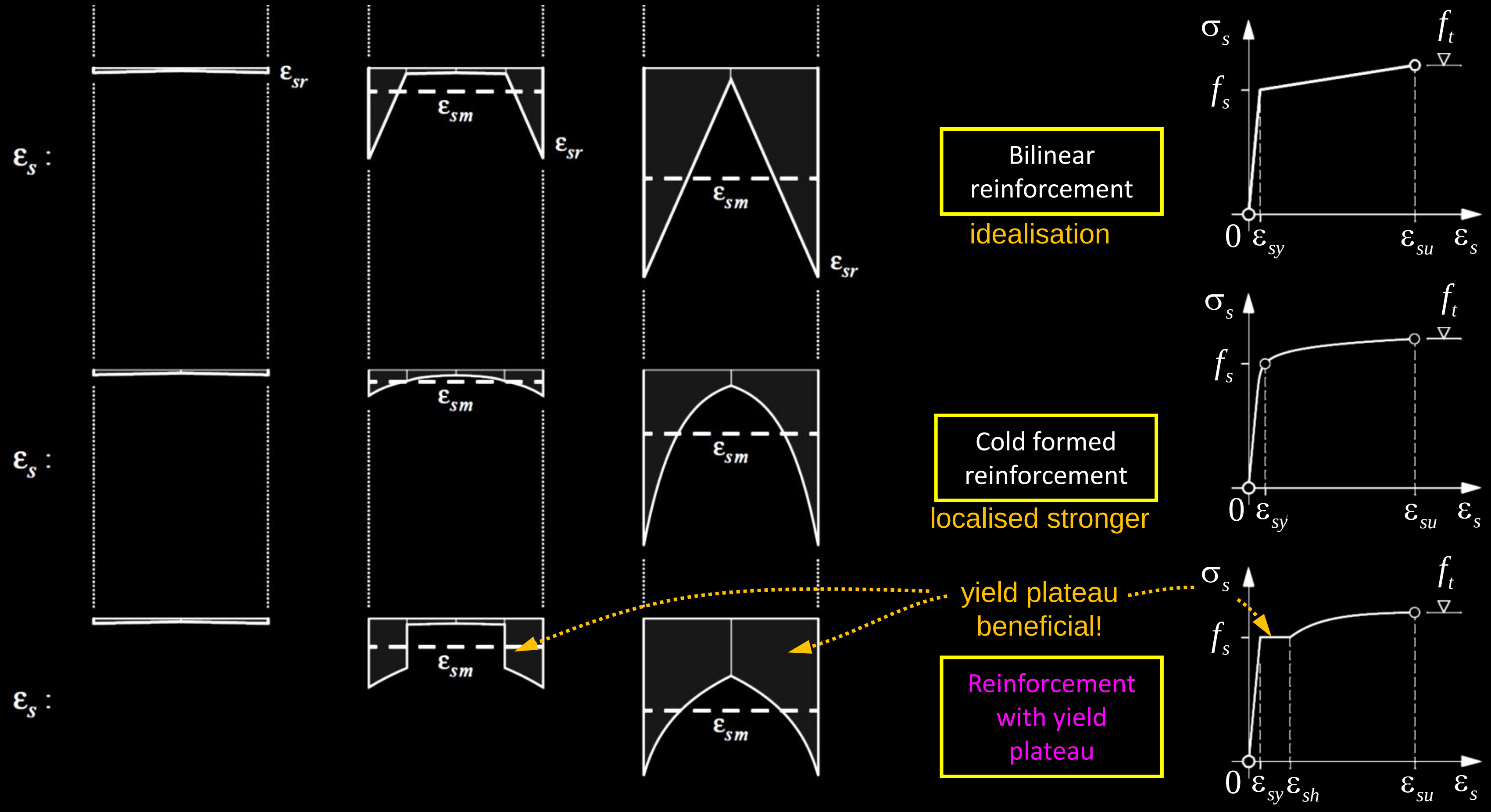
Additional considerations: influence of the reinforcement hardening properties

1 Reinforcement elastic over entire length

2 Reinforcement yielded near cracks

3 Reinforcement yielded over entire length

Strains in the three regimes for the following characteristic curves of reinforcement:

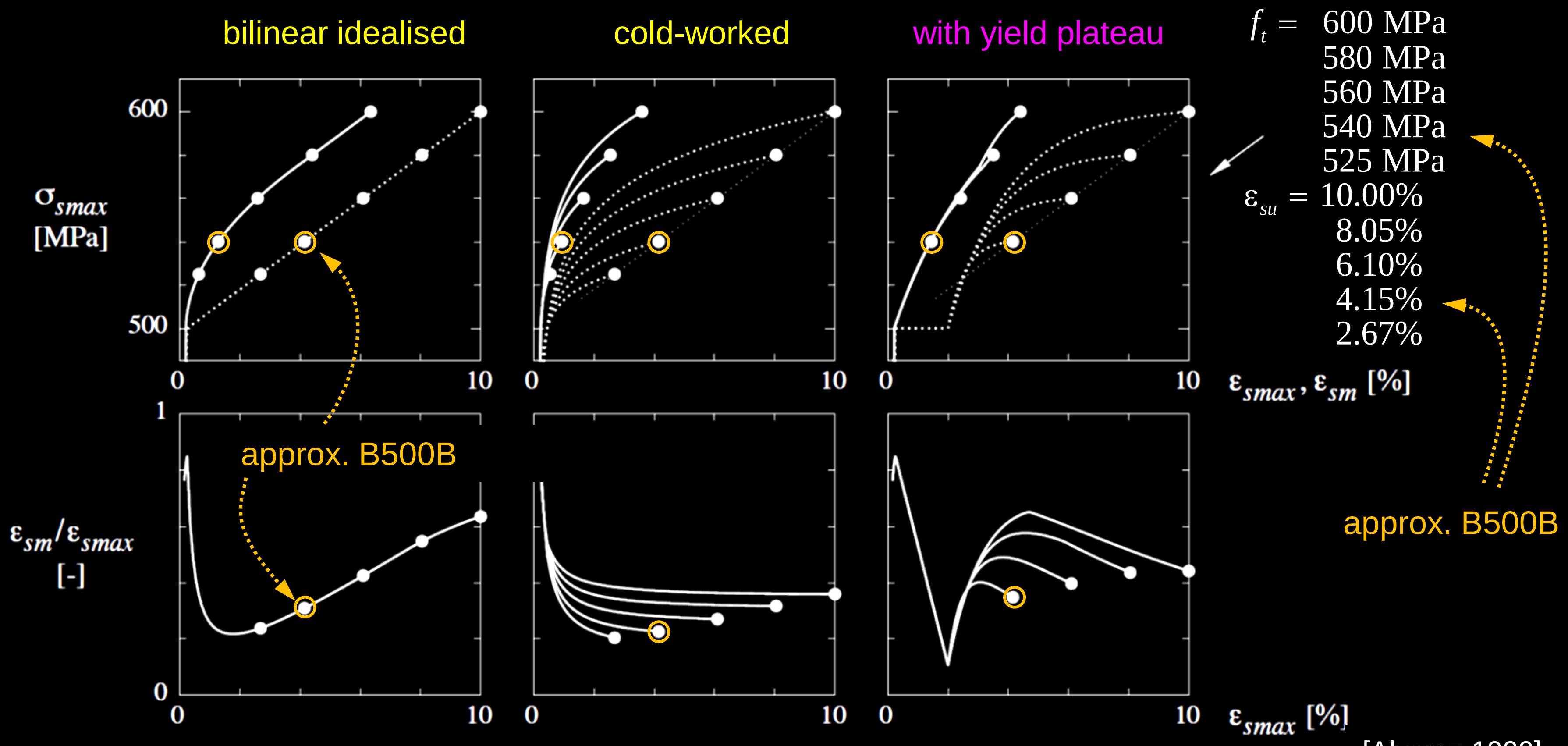


Beams – Deformation capacity

Additional considerations: influence of the reinforcement hardening properties

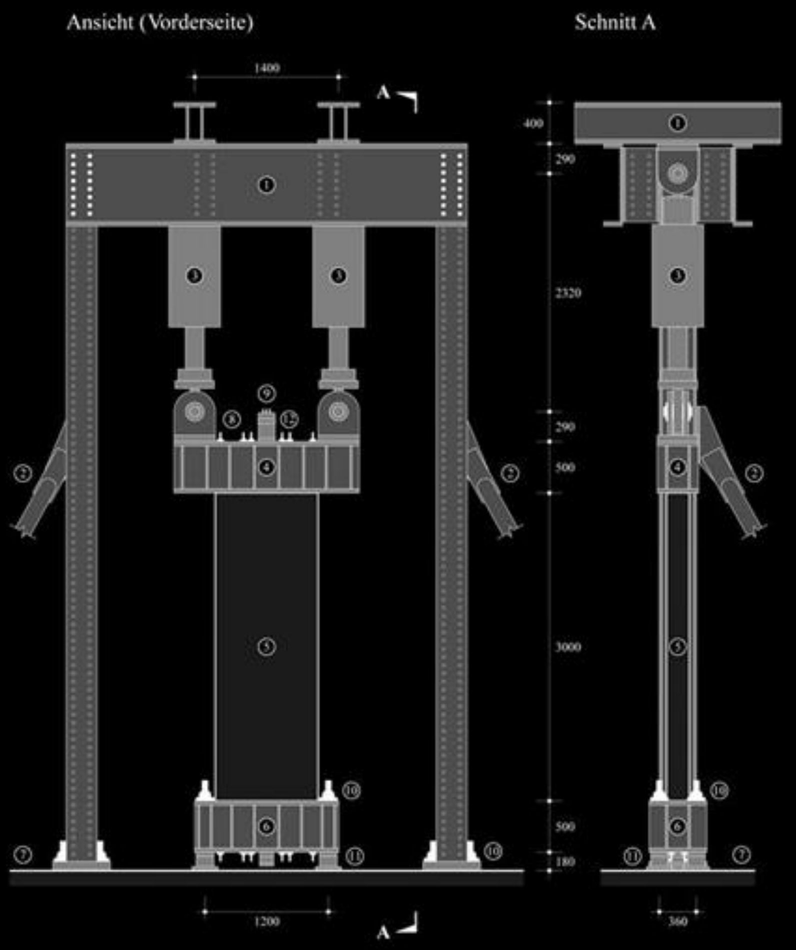
→ Reinforcement with a yield plateau is more favourable than cold-formed reinforcement, especially in case of failure in regime 2 (yield plateau contributes as an "additional" strain over the entire yielded area)

→ The bilinear idealization overestimates the deformation capacity for a reinforcement with high ductility



[Alvarez 1999]

Tension experiments – Dr. M. Alvarez: Test setup



① Stahl-Reaktionsrahmen
② Druckstreben (Knicksicherung)
③ Servohydraulische Pressen (2 MN Kapazität)
④ Oberer Krafteinleitungsträger
⑤ Versuchskörper
⑥ Unterer Krafteinleitungsträger
⑦ Aufspannboden
⑧ Verankerungen der Anschlussstäbe
⑨ Verankerung des Vorspannkabels
⑩ Vorspannstangen (Verankerungen im Aufspannboden)
⑪ Kraftmessdosen (1 MN Nennbereich)
⑫ Kraftmessdose (2 MN Nennbereich)

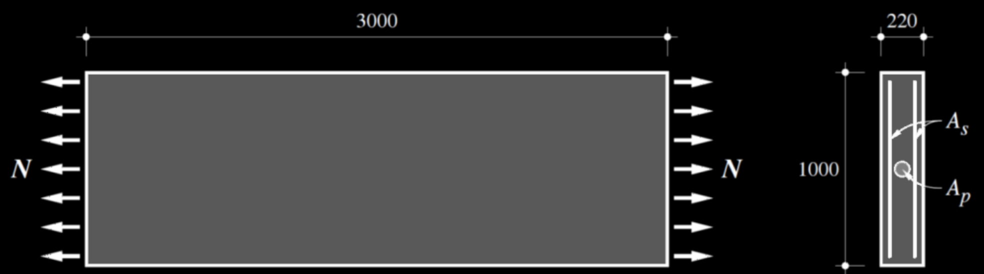
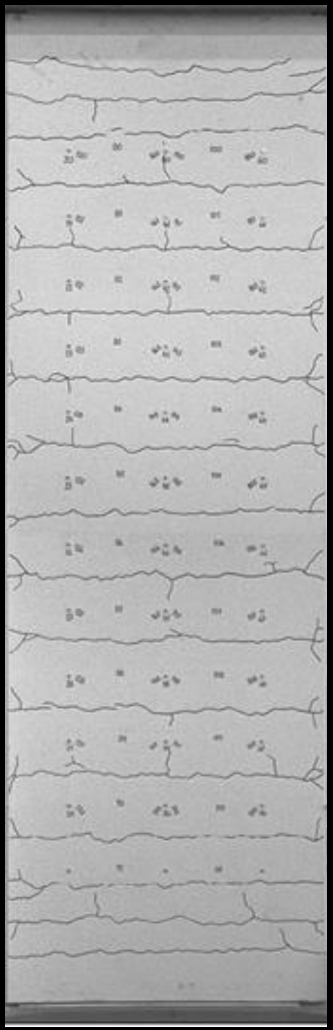


Bild 1.1 – Konzept der Zugversuche, [mm].

Versuchskörper	Z1	Z2	Z3	Z4	Z5	Z6	Z7	Z8	Z9
Längsbewehrung A_s [mm^2]	2156					1232	616	1540	4312
Vorspannbewehrung A_p [mm^2]	0					1050		0	
Betonstahlqualität	H			N	L	*H*			
Bügelbewehrung [mm]	Ø 8 @ 200		0	Ø 8 @ 200					
Würfeldruckfestigkeit des Betons f_{cw} [MPa]	50	90	50						

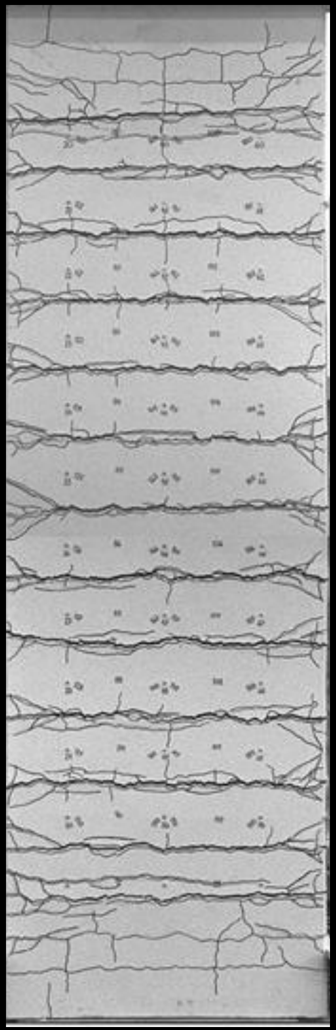
H: $\varepsilon_{su} = 14.6\%$ $f_t/f_s = 1.26$
N: $\varepsilon_{su} = 3.8\%$ $f_t/f_s = 1.05$
L: $\varepsilon_{su} = 3.1\%$ $f_t/f_s = 1.06$

Tension experiments – Dr. M. Alvarez: Crack patterns at failure



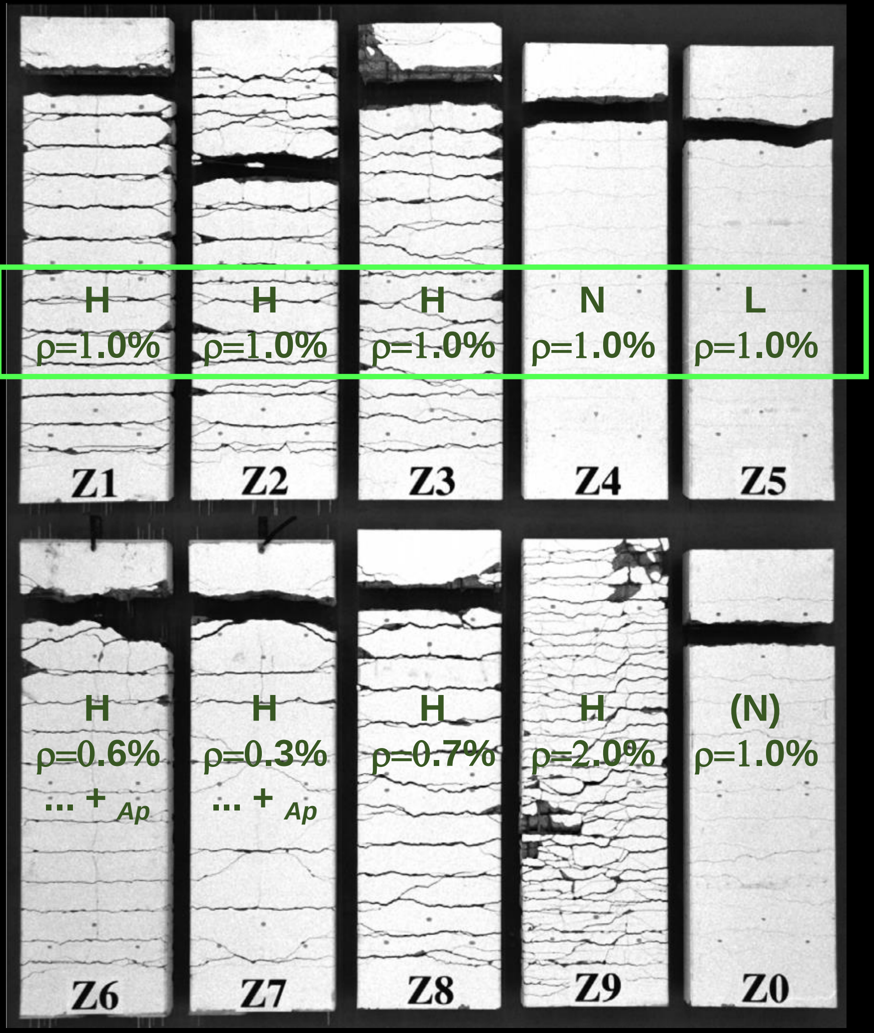
Experiment Z1 - LS

Yielding



Experiment Z1 - LS

Hardening

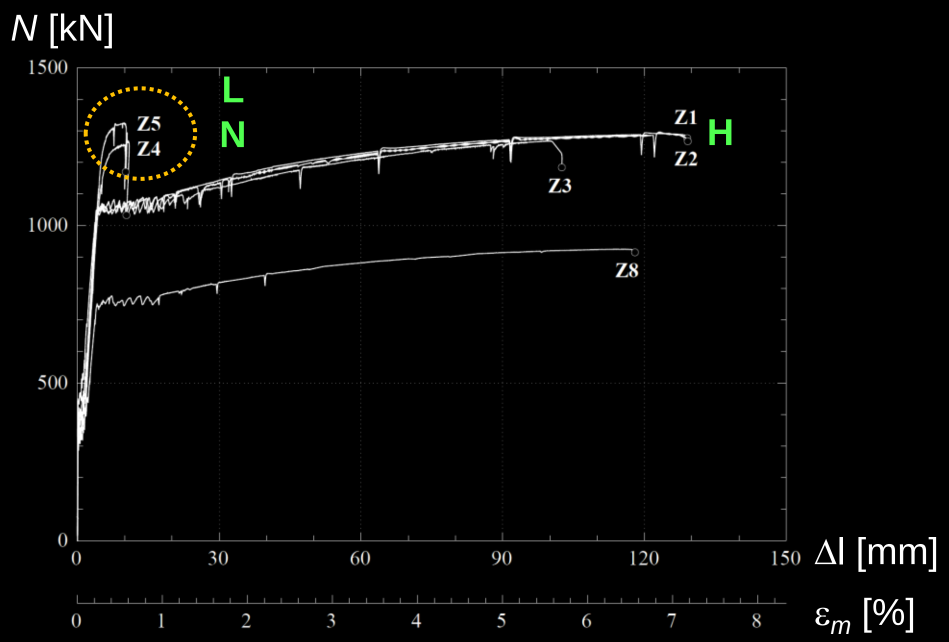


Specimens after failure: plastic (=remaining) deformations differ strongly

Tension experiments – Dr. M. Alvarez: Test results

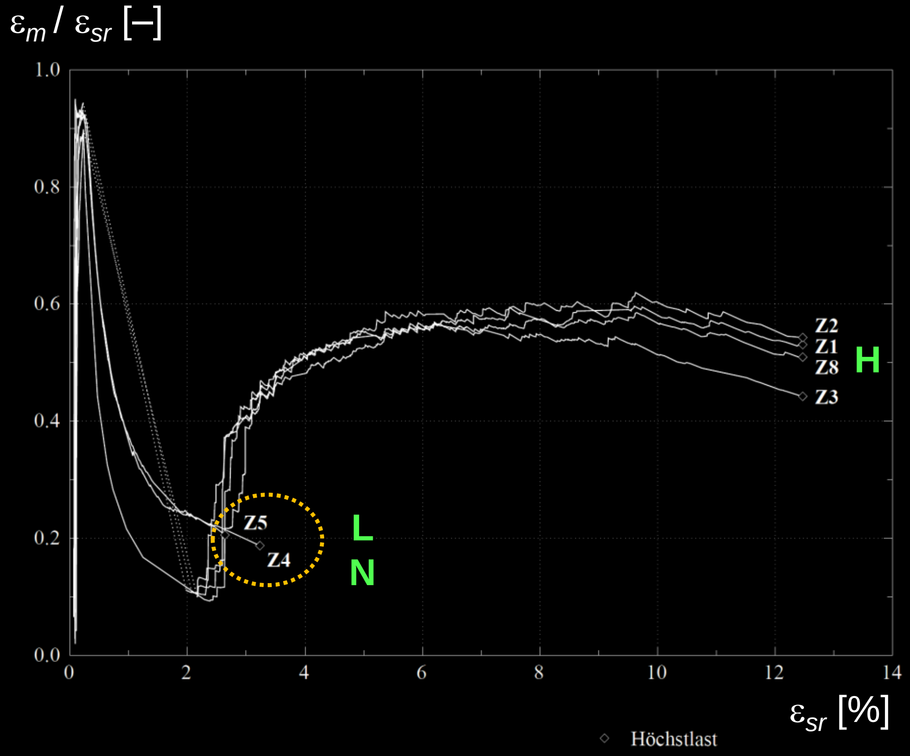
Load-deformation behaviour considering bond

→ Deformation capacity severely impaired for reinforcement with low ductility (failure deformation and hardening!)



Ratio of average elongation to maximum elongation at the cracks considering bond

→ Good agreement with tension chord model (almost identical if the real bare steel curve is taken into account)



Deformation capacity

Summary

- The concrete strength should be reduced in plastic analysis depending on the cracking state of the structure and on the material brittleness.
- The concrete contribution in tension between two cracks stiffens the response of bonded reinforcement with respect to bare (unbonded) reinforcement. This tension-stiffening effect affects the serviceability response of the structure but also reduces the deformation capacity of the reinforcement. Assuming simplified bond relationships (as e.g. in the Tension Chord Model) is sufficient for modelling tension-stiffening.
- *Deformation capacity* and *deformation demand* are coupled. The interaction can only be neglected for moderate redistributions of the internal forces.
- The *deformation demand* can be determined approximately with reasonable effort using simplified assumptions (constant bending stiffness of the elastic areas, rigid-ideal plastic M-Θ *relationships of* the plastic hinges).
- Even with complex calculations, the *deformation capacity* can only be roughly estimated because it depends on several effects and assumptions that cannot be precisely quantified:
  - Bond behaviour, in particular, crack spacing
  - Mechanical properties of the reinforcement (hardening ratio and deformation of failure, with or without yield plateau)
  - Force flow in the area of plastic hinges, in particular, variation of the force in the tension chord (→ the mean deformations averaged over the length of the plastic hinge are smaller than the mean deformation of a tension chord under constant tensile force!)
- In practice, it is therefore advisable to avoid the verification of the deformation capacity for new structures whenever possible (complying with the condition $x/d < 0.35$). Otherwise, it is often easier to ignore the redistribution of internal forces, i.e. to verify the structural safety for the elastic stresses including restraint stresses (even if the estimation of the restraints is also time-consuming and requires assumptions).
- If the deformation capacity needs to be verified (e.g. for existing structures), engineering judgement must be applied. The decisive parameters should be accounted for as accurately as possible (reinforcement: determine hardening characteristics, not just f_s).