1. Introduction

Learning objectives

Based on this chapter, the students are able to:

- understand the relevance of deepening their knowledge of the load-bearing behaviour of concrete structures to
  - conceive and design efficient new structures hence reduce greenhouse gas emissions
  - pertinently assess existing structures hence extend their lifespan with minimum intervention

- identify and distinguish different methods to *analyse and design* concrete structures:
  - describe the differences between elastic solutions and plastic solutions (limit analysis methods).
  - differentiate between approaches to design a new structure and to assess an existing one.
  - explain the theorems of limit analysis, the underlying assumptions and their consequences on structural design practice.

Why Advanced Structural Concrete?

Impact of structural engineer in a design office

Source: The Institution of Structural Engineers – How to calculate embodied carbon

Holistic environmental sustainability

reduce emissions of materials

design efficient structures

reduce comfort level

$$\text{annual emissions per use} = \frac{\frac{\text{emissions}}{\text{mass}} \cdot \frac{\text{mass}}{\text{floor area}} \cdot \frac{\text{floor area}}{\text{use}}}{\text{service life}}$$

drastic reduction required

societal needs / demands

ensure durability / extend lifespan

[Gebhard 2023, based on Flatt and Wangler 2022]

Holistic environmental sustainability

requires profound knowledge beyond the fundamentals taught in BSc courses

design efficient structures

$$annual\ emissions\ per\ use = \frac{\dfrac{emissions}{mass} \cdot \dfrac{mass}{floor\ area} \cdot \dfrac{floor\ area}{use}}{service\ life}$$

ensure durability / extend lifespan without interventions

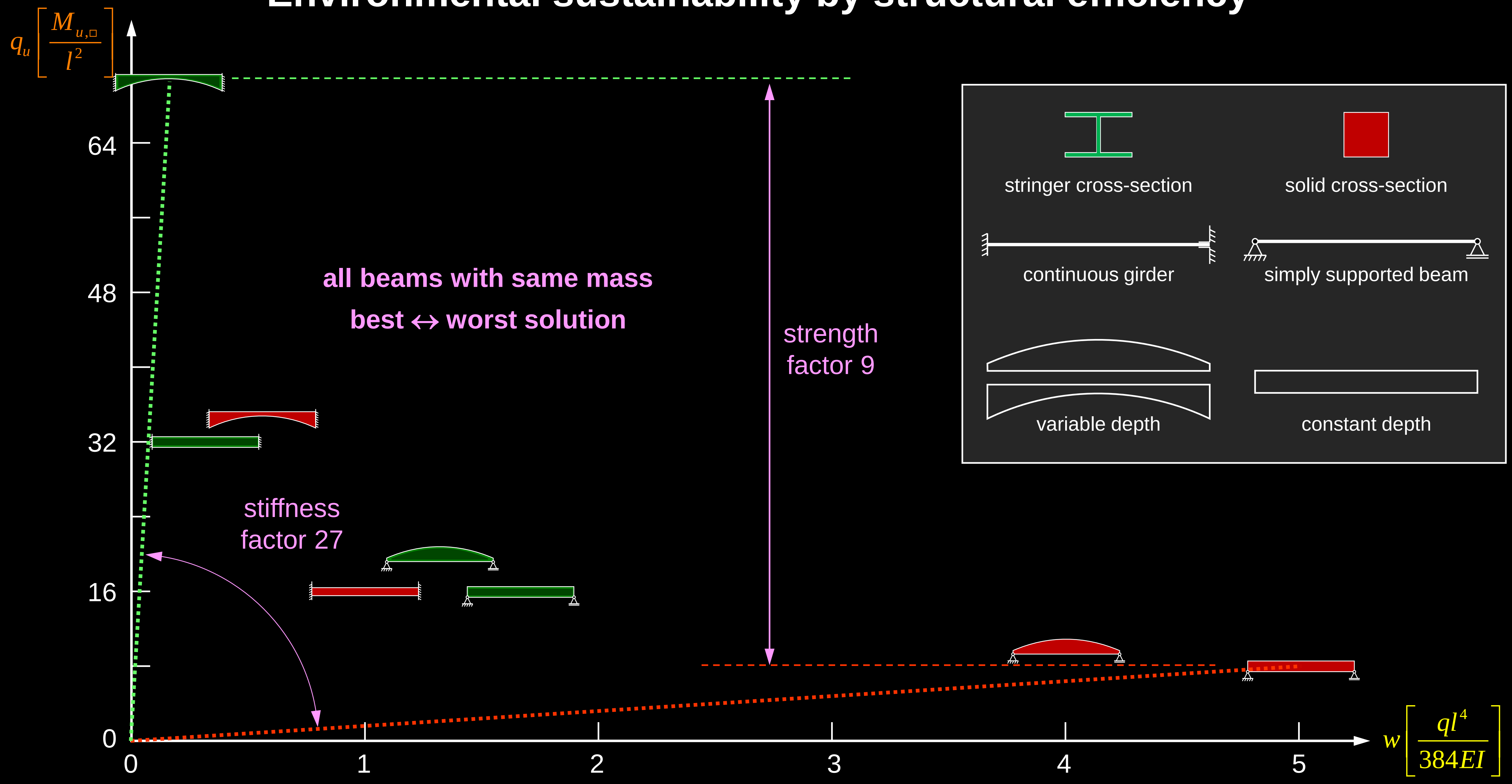
[Gebhard 2023, based on Flatt and Wangler 2022]

Environmental sustainability by structural efficiency


Gatti Wool Factory – Pier Luigi Nervi, 1951


Ausgleichsbecken Les Marécottes - A. Sarrasin - 1926

Environmental sustainability by structural efficiency

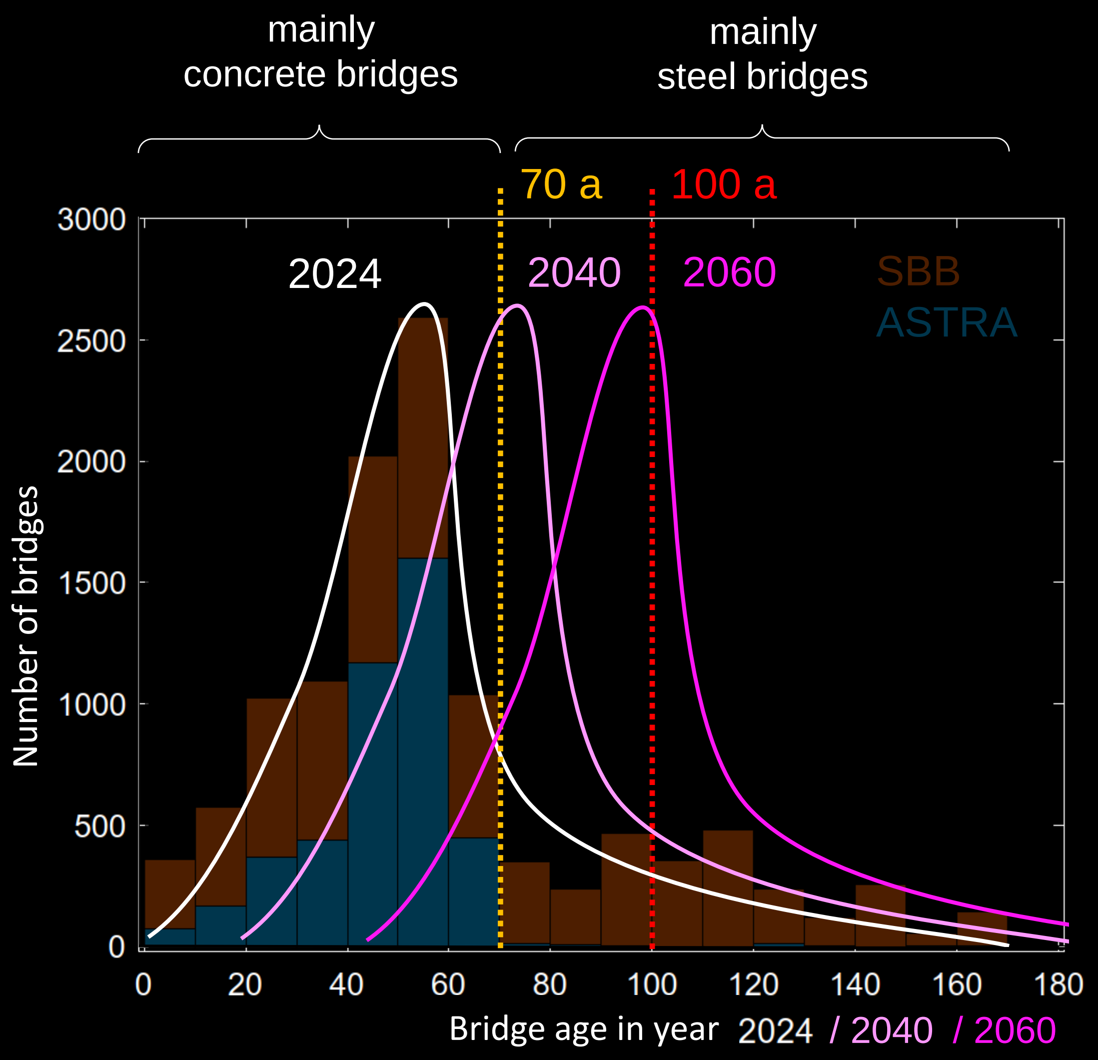


Environmental sustainability by structural efficiency


Manoir Haute Roche – Hennebique, 1914

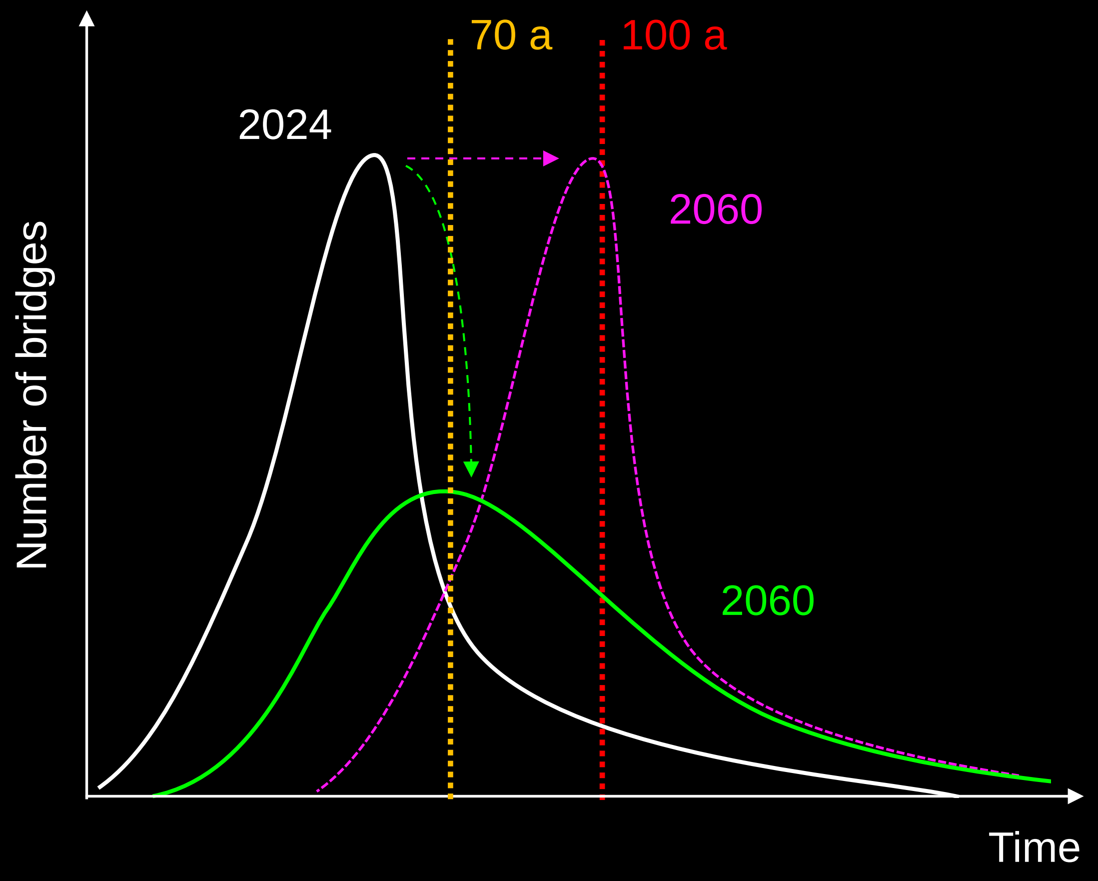

Teufelsschluchtbrücke – Solothurn

Stretching the lifespan of existing (concrete) structures



Many bridges are reaching their planned service life
Massive number of existing structures
(CH: 40’000, Europe 1 Mio, USA 650’000, …)

Stretching the lifespan of existing (concrete) structures



Solution:
maximise service life with
minimum interventions

- without excessive / unacceptable risk
- with limited resources

→ pertinent assessment of structural safety is key

Structural safety		Reality: not ok	Reality: ok
Analysis	not ok	ok	resources
Analysis	ok	risk of collapse reputation	ok

Dimensioning of new structures

Ensure ductility by conceptual measures

→ neglect concrete tensile strength and provide minimum reinforcement to avoid brittle failure at crack formation

→ ensure failure by yielding of reinforcement
… limit reinforcement ratio (e.g. $x/d < 0.35$ in bending)
… use conservative value of concrete compressive strength

→ verify deformation capacity (rarely in new design)

Design based on lower-bound theorem of plasticity theory

→ define and consistently follow desired load path

→ structural elements have clearly defined functions (e.g. web with pure in-plane loading)

→ conservative design

"The engineer tells the structure how to carry the loads"

~~"The structure tells the engineer what to calculate"~~

Simple models sufficient

Restraint stresses can be neglected

Redundancy and robustness

Solving problems conceptually

Designing instead of calculating

Structural Concrete I/II

BSc degree

Structural assessment of existing structures

Maximise service life with minimum interventions

→ avoid strengthening and retrofitting

→ without excessive / unacceptable risk

Challenges

→ dimensions and reinforcement are given, cannot be designed for a chosen load path

→ effect of deterioration (corrosion, ASR, …)?

→ ductility often not given a priori (lacking minimum reinforcement, heavy prestressing with $x/d > 0.35$, …)

→ structural elements with several functions (e.g. web with combined in-plane and out-of-plane loading)

→ Evaluate the deformation capacity, load-deformation behaviour is relevant

⇒

Simple models often insufficient (not applicable or overly conservative)

Use of updated material properties

Verification of deformation capacity required

Numerical approaches required to exploit load-bearing capacity

More demanding than design of new structures!

Advanced Structural Concrete

MSc, Major in Structural Engineering

Methods of analysis and design

Structural analysis and design – Linear elastic (FE) analyses

Actions
static boundary conditions

Materials
elastic $\sigma_i = E_i \cdot \varepsilon_i$

Static system
kinematic boundary conditions

↓

Equilibrium conditions **+** Kinematic relationships (compatibility)

↓

Linear elastic solutions

Approximations of the **stress state** (e.g. stress fields)

too soft → **minimise complementary energy** (find stiffest (*) solution)

→ **stress state**

“Exact” (*) solution (e.g. frame analyses, analytical solutions)

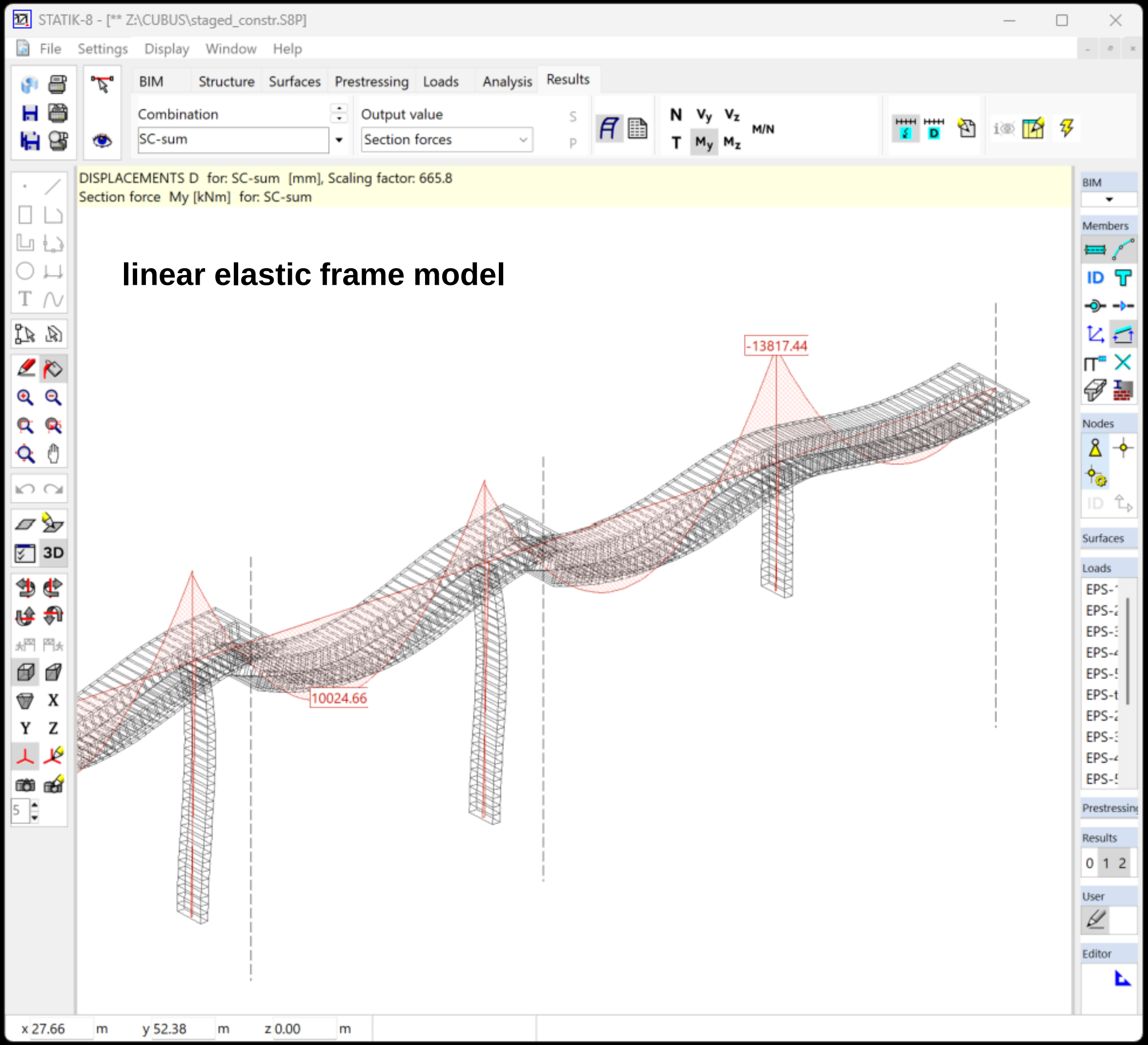
(*) residual stresses, restraints, load history, cracking, nonlinearity?

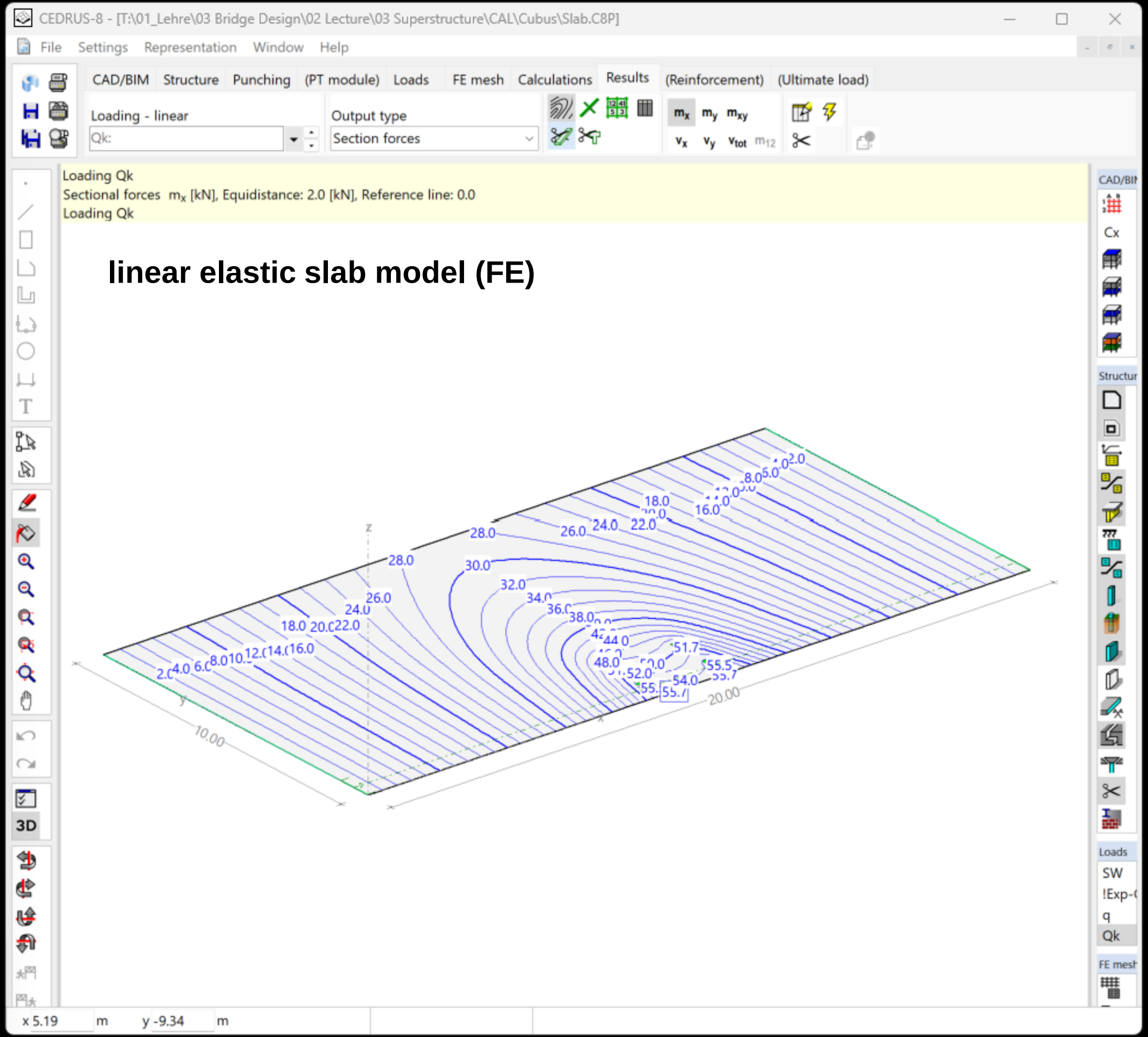
Approximations of the **deformation state** (e.g. linear FE)

too stiff → **minimise potential energy** (find softest (*) solution)

→ **strain state**

Structural analysis and design – Linear elastic (FE) analyses





[STATIK / CEDRUS software (CUBUS AG, Zürich)]

Structural analysis and design – Limit analysis methods (i)

Actions
static boundary conditions

Materials
yield strength

Static system
support reactions

↓

Equilibrium conditions

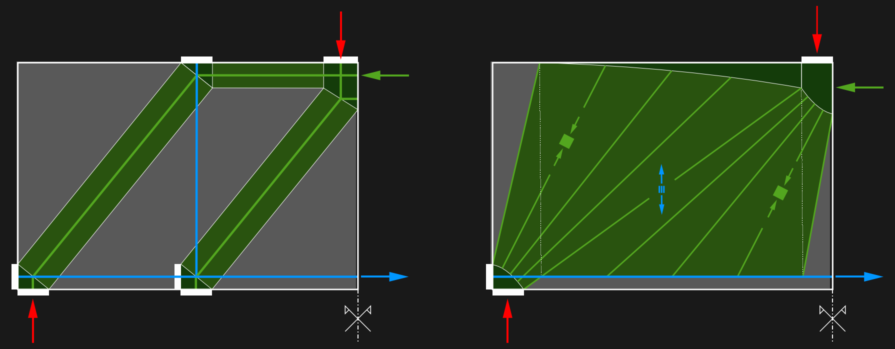
↓

Limit analysis – Lower-bound solutions

Walls: stress fields, strut-and-tie models

Slabs: strip method

conservative → **maximise ultimate load**



Results of linear elastic FE analyses

← stress state / ultimate load

Structural analysis and design – Limit analysis methods (ii)

Actions

loads ($\rightarrow$ external work)

Materials

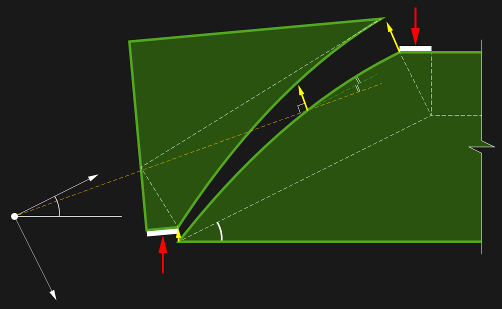
yield strength ($\rightarrow$ dissipation)

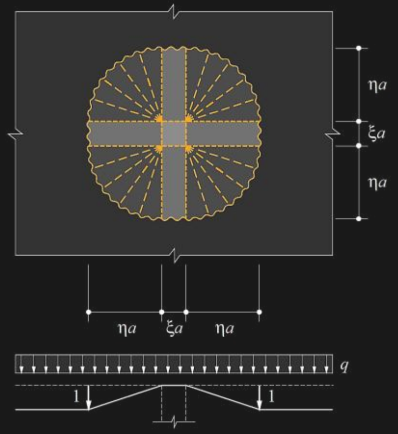
Static system

kinematic boundary conditions

Simplified kinematic relationships (mechanism)

Limit analysis – Upper-bound solutions





Approximations of the **deformation state = kin. admissible collapse mechanisms**

unconservative $\rightarrow$ **minimise ultimate load**

collapse mechanism / ultimate load

Structural analysis and design – General NLFE analyses

Actions
static boundary conditions

Materials
general $\sigma_i = \sigma(\vec{\varepsilon})$

Static system
kinematic boundary conditions

↓

Equilibrium conditions + Kinematic relationships (compatibility)

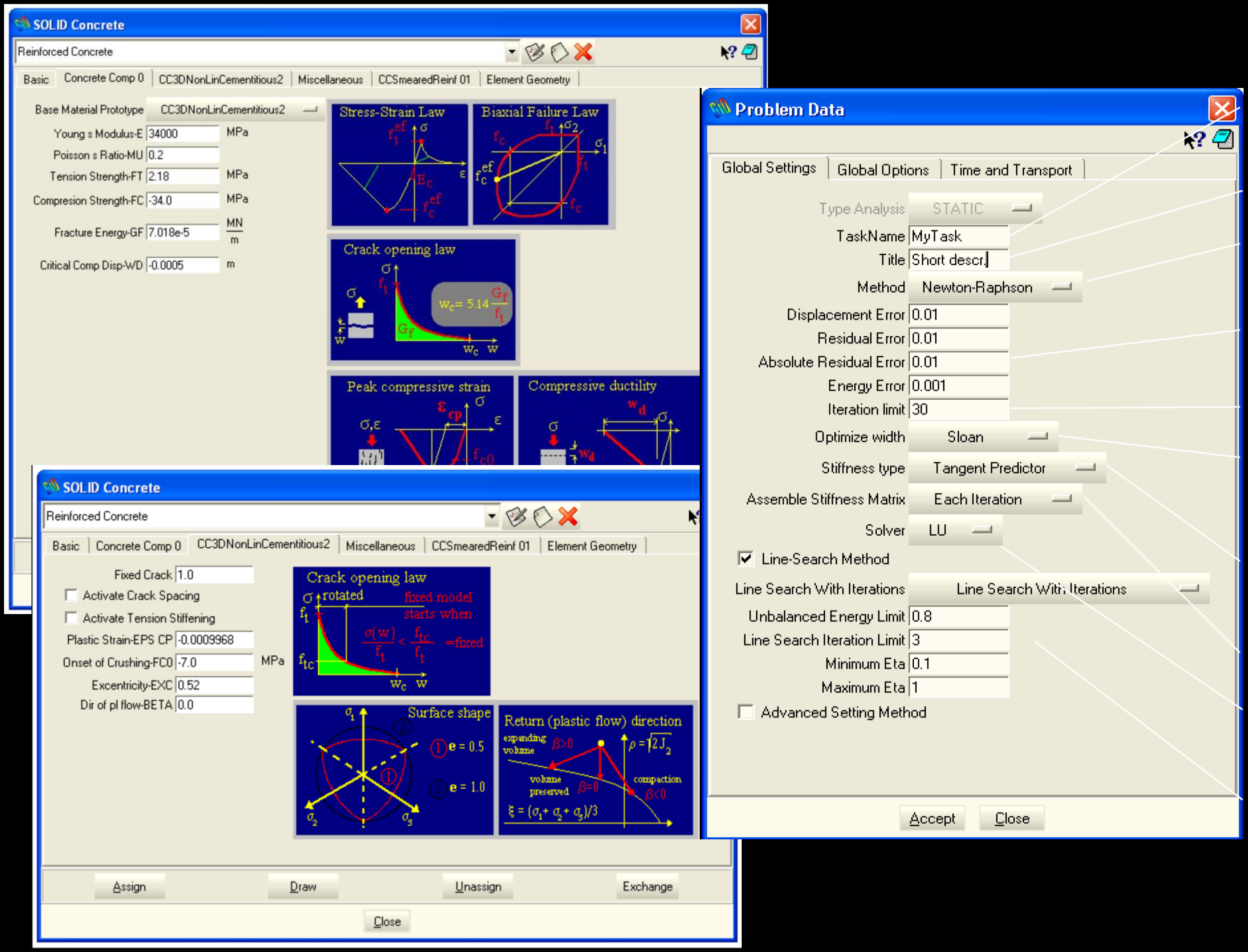
↓

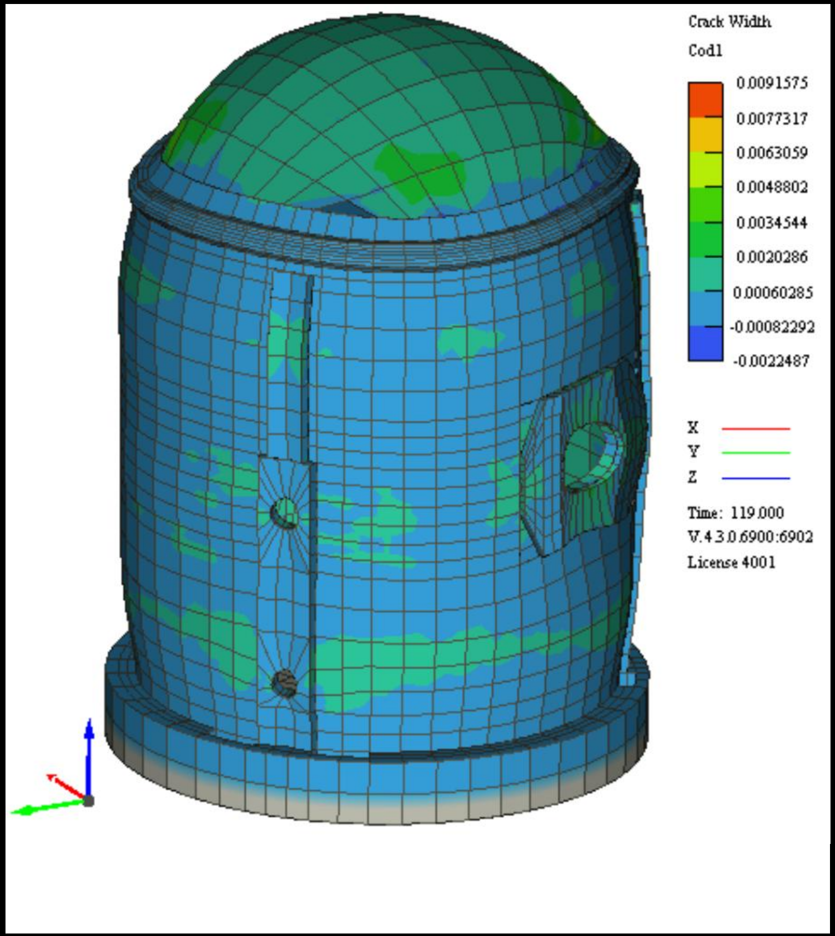
General nonlinear analyses (NLFE e.g. Atena, Diana, Abaqus, XFEM)

Realistic pre-/postdiction(*) of load-deformation behaviour
… activation of plastic reserve capacities
… verification of deformation capacity
… investigation of serviceability (crack widths)
(*) many non-standard (material) parameters, requiring calibration / tuning (e.g. fracture energy, flow direction, shear retention factor, …)
requiring specific expertise, very time consuming → not suitable for industry

stress state ← → strain state

Structural analysis and design – General NLFE analyses





[ATENA NLFE software (Cervenka Consulting)]

Structural analysis and design – Lean RC-specific NLFE analyses

Actions

static boundary conditions

Materials

standard $\sigma_i = \sigma(\vec{\varepsilon})$, $f_{ct} = 0$

Static system

kinematic boundary conditions

↓

Equilibrium conditions

+

Simplified kinematic relationships (e.g. $\nu = 0$)

↓

Lean RC-specific NLFE analyses (e.g. EPSF, CSFM, CMM-U, FELA)

Realistic prediction of load-deformation behaviour

… activation of plastic reserve capacities

… verification of deformation capacity

… investigation of serviceability (crack widths)

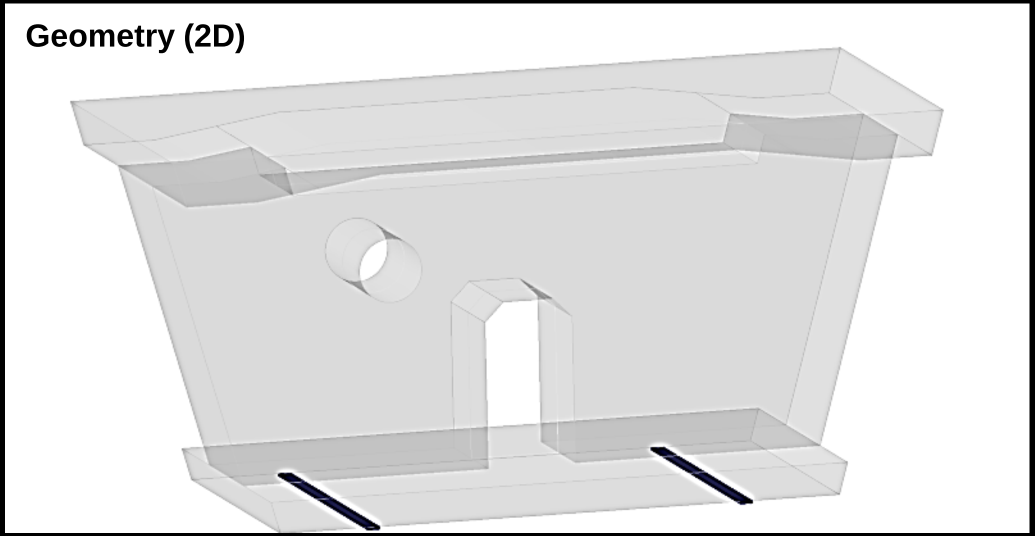
only standard material parameters (f_c, f_{sy}, f_{su}, ε_{su}), no calibration required

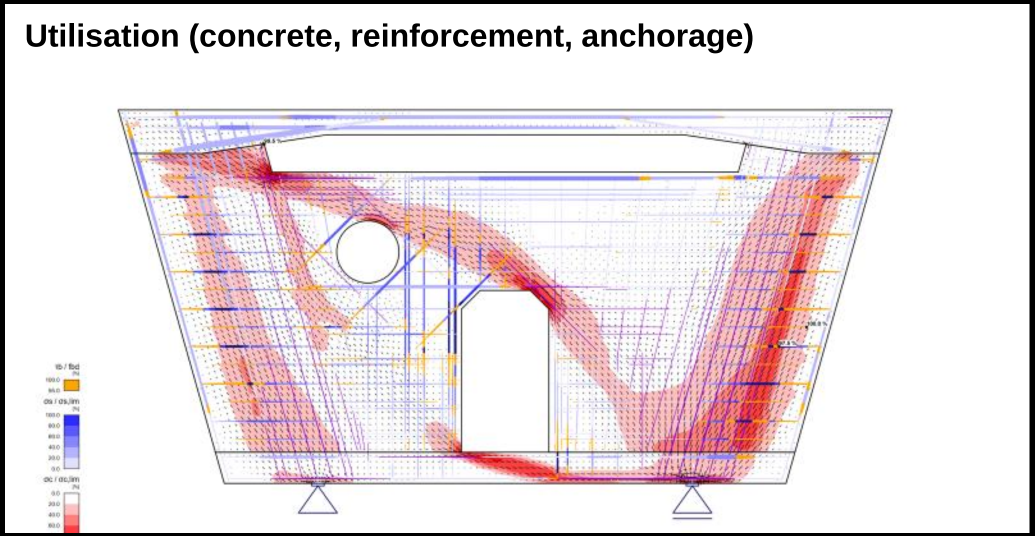
requiring expertise, time consuming $\rightarrow$ suitable for structural assessment

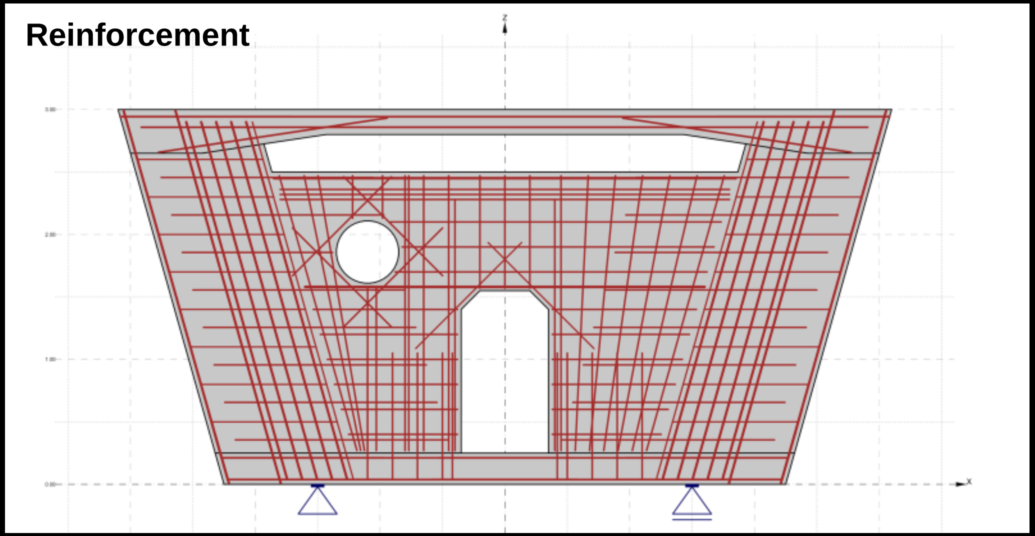
← stress state

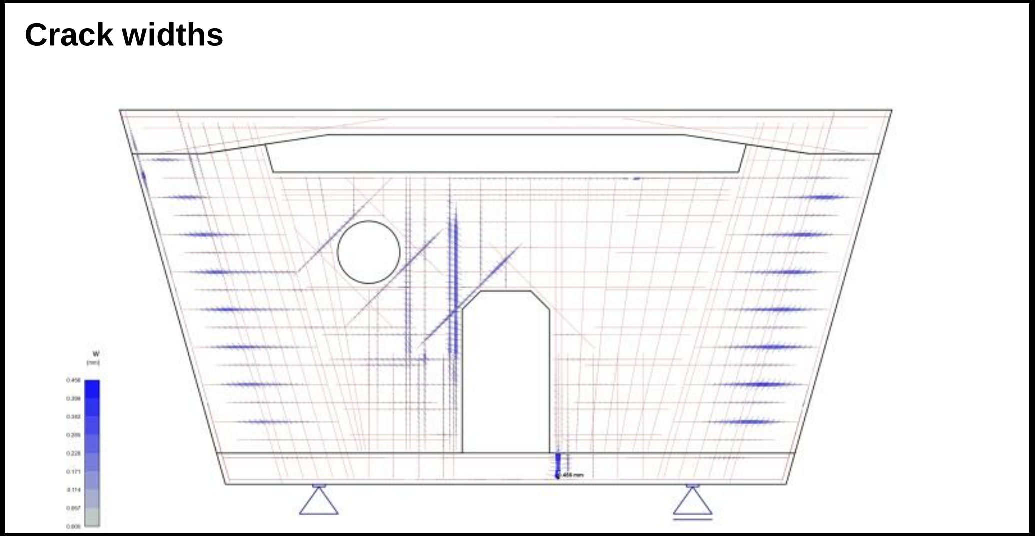
→ strain state

Structural analysis and design – Lean RC-specific NLFE analyses



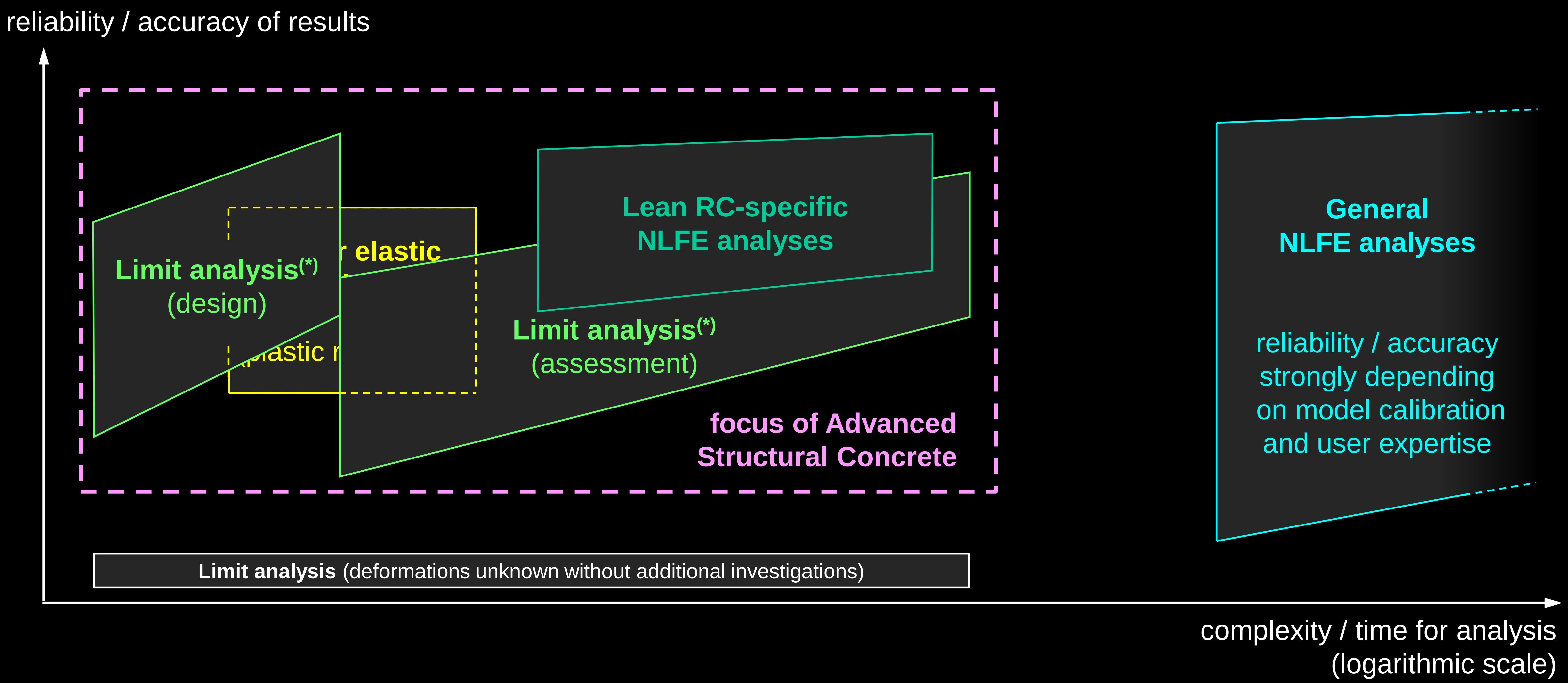






CSFM (Compatible Stress Field Method, ETHZ / IdeaStatica Detail)
[Davide Bianchi, dsp Ingenieure + Planer AG, 2025]

Methods of analysis and design (schematic)



(*) deformations may be obtained by complementing lower-bound solutions with pertinent member stiffnesses, e.g. based on the Tension Chord Model

Fundamentals of limit analysis methods

Theory of plasticity – Limit analysis

Lower-bound (static) theorem

Every loading for which it is possible to specify a statically admissible stress state that does not infringe the yield condition is not greater than the limit (ultimate) load.

(a statically admissible stress state must satisfy equilibrium and the static boundary conditions)

Upper-bound (kinematic) theorem

Every loading that results from equating the work of external forces for a kinematically admissible deformation state with the associated dissipation work is not less than the limit (ultimate) load.

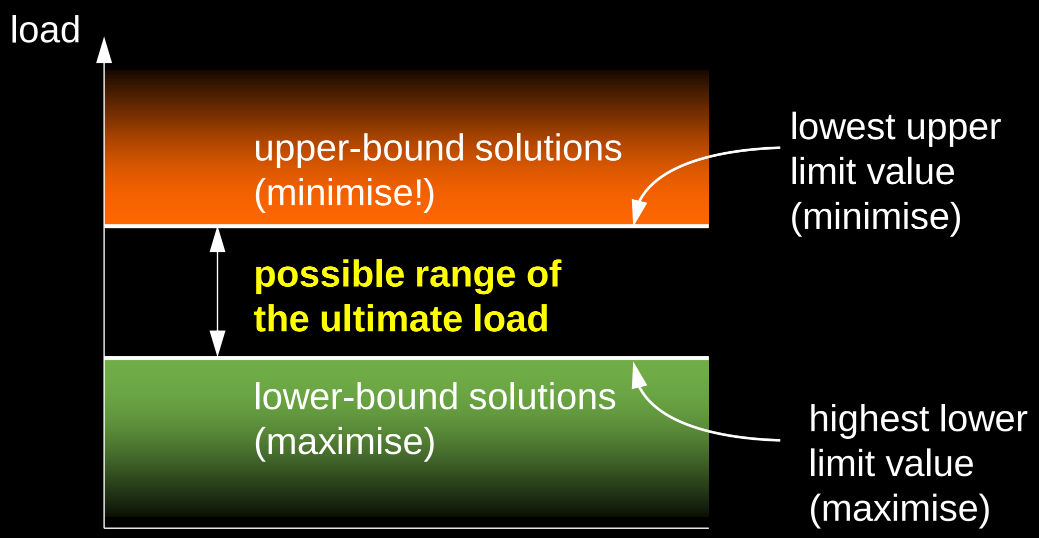
(kinematically admissible: kinematic relationships and kinematic boundary conditions are fulfilled)

Compatibility theorem

A load for which a statically admissible stress state that does not infringe the yield condition and a compatible kinematically admissible state of deformation can be specified is a limit (ultimate) load

The force and deformation states linked by this theorem constitute a *complete solution* to the respective problem.

NB. If upper- and lower-bound solution coincide, the ultimate load has been found – no need to verify compatibility, see figure.



Theory of plasticity – Limit analysis

Prerequisites of the theorems of limit analysis

Strictly, the theorems of limit analysis are valid for perfectly plastic behaviour with:

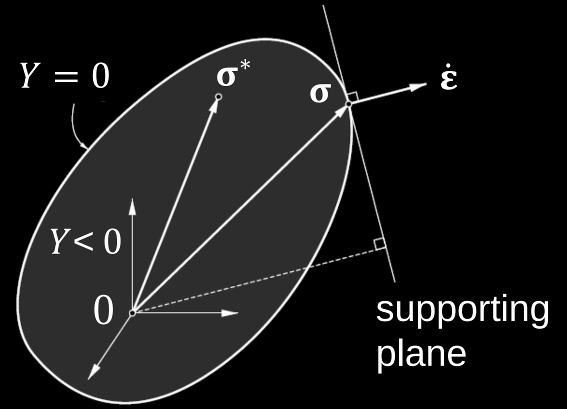
- a (weakly) convex yield surface $Y(\boldsymbol{\sigma}) = 0$ and
- an associated flow rule:
  $\dot{\boldsymbol{\varepsilon}} = \kappa \operatorname{grad} Y \; (Y = 0{:}\, \kappa \geq 0;\, Y < 0{:}\, \dot{\boldsymbol{\varepsilon}} = 0)$

The latter stipulates that (i) plastic strain increments $\dot{\boldsymbol{\varepsilon}}$ occur only for stress states on the yield surface (rigid-perfectly plastic behaviour) and (ii) these strain increments $\dot{\boldsymbol{\varepsilon}}$ are orthogonal to the yield surface Y.

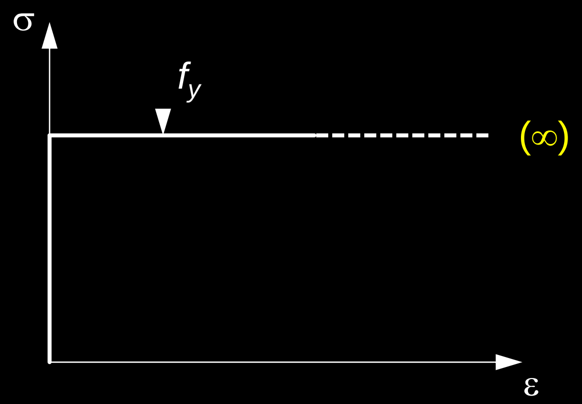
The two conditions of convexity and orthogonality are equivalent to the principle of maximum dissipation:

→ if the yield surface Y is (weakly) convex, any plastic strain increment $\dot{\varepsilon}$ generates its maximum dissipation (scalar product $\mathrm{d}D = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}$) for the stress state(s) where it is orthogonal to Y (see top left figure)

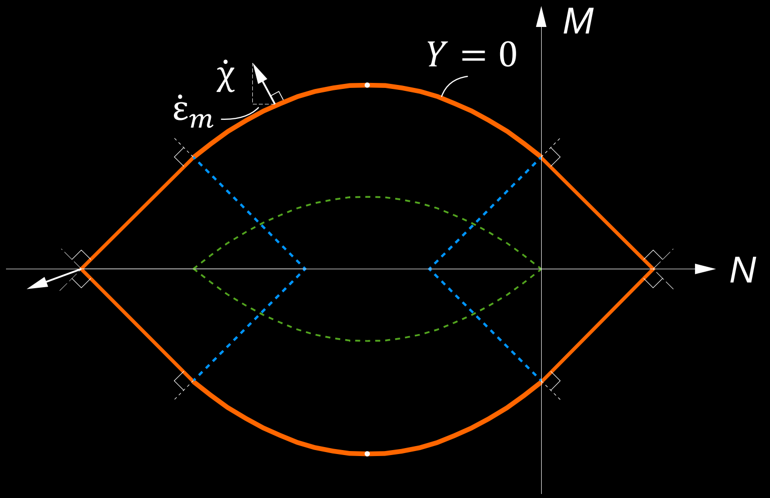
convexity + orthogonality = maximum dissipation:

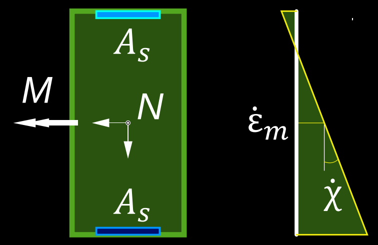


assumed material behaviour: perfectly plastic



example: M-N interaction (generalised stresses {M,N} and strains {$\dot{\varepsilon}_m$, $\dot{\chi}$}):
(see BSc lecture Stahlbeton I)





$$Y_c = \pm M_{yc} + N_c\left(\frac{h}{2} + \frac{N_c}{2bf_c}\right) = 0$$

$$\pm\frac{\dot{\varepsilon}_m}{\dot{\chi}} = \frac{h}{2} + \frac{N_c}{bf_c} = \frac{\partial Y_c / \partial N_c}{\partial Y_c / \partial M_{yc}}$$

Theory of plasticity – Limit analysis

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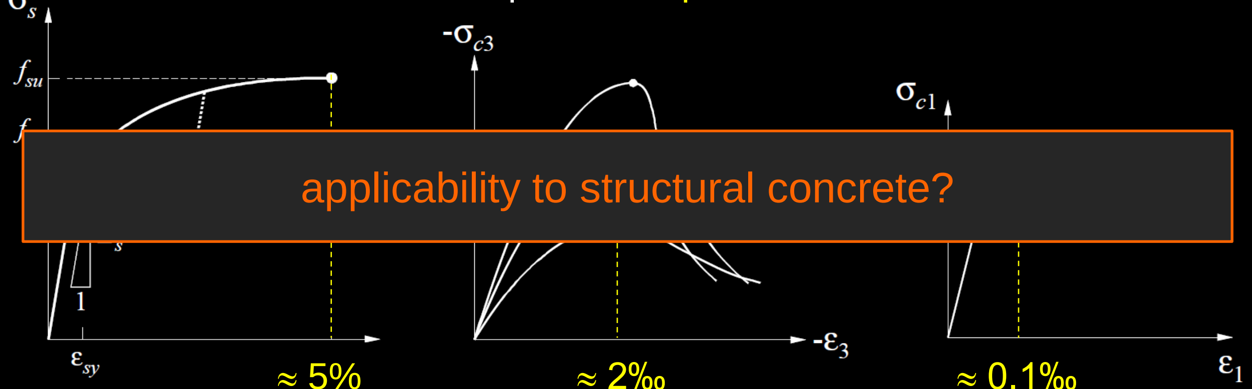
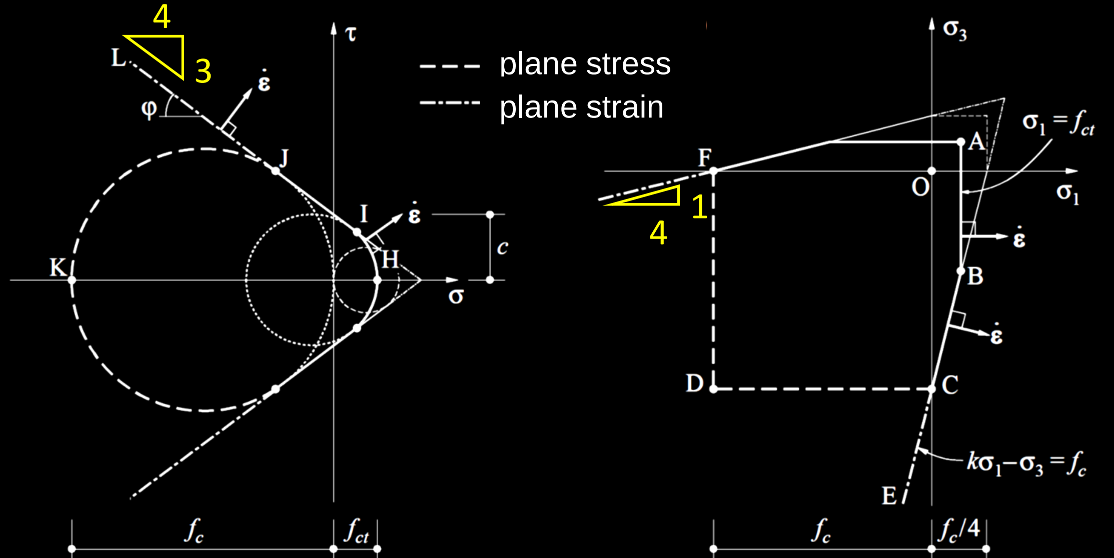
reinforcement: ductile | concrete in uniaxial compression: quasi-brittle | concrete in tension: brittle



applicability to structural concrete?

common yield criterion for concrete: Modified Coulomb with tension cut-off ($\tan\varphi = 0.75 \rightarrow c = f_c/4$, $\varphi \approx 37°$, usually $f_{ct} = 0$)



Theory of plasticity – Limit analysis

Prerequisites of the theorems of limit analysis

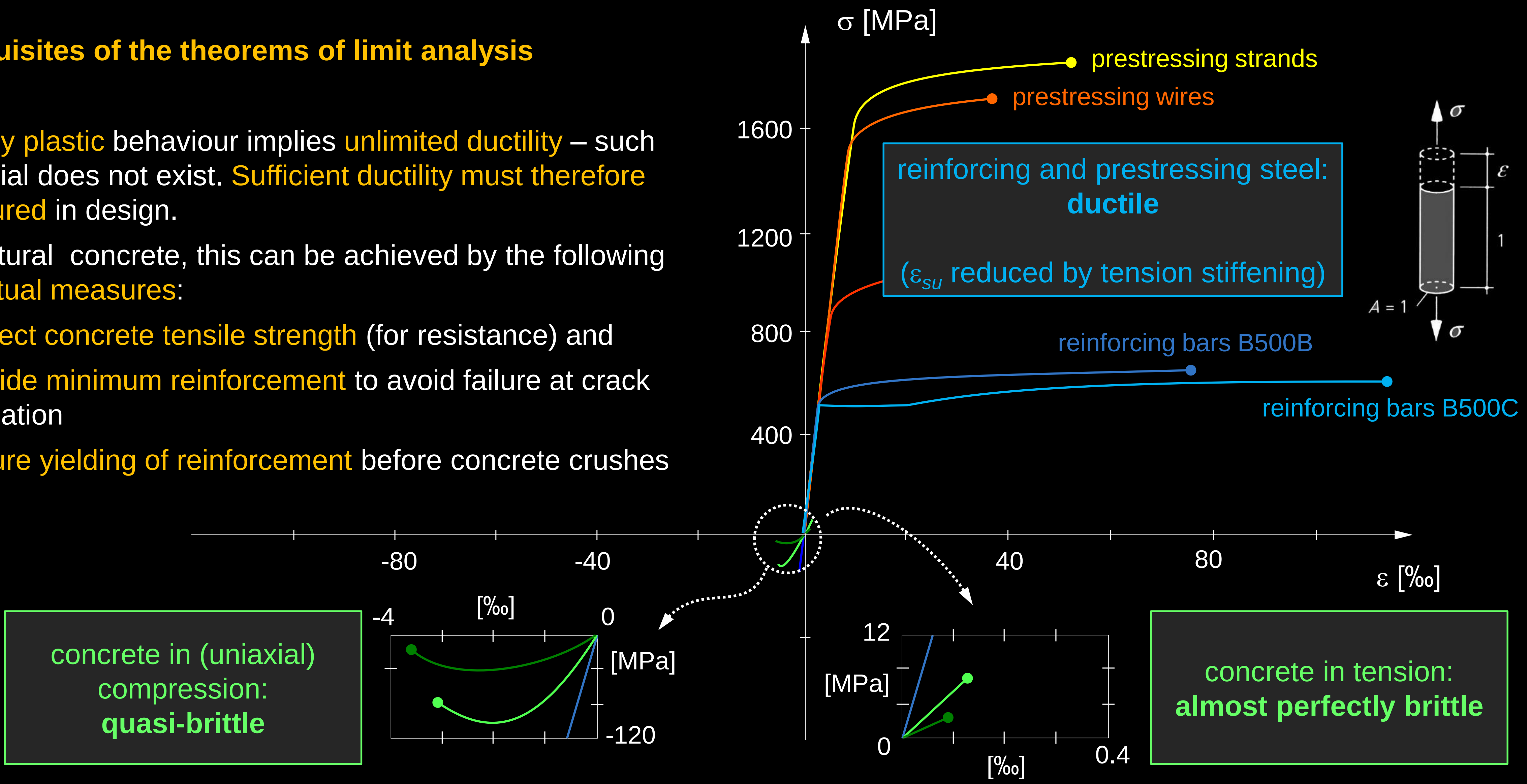
Perfectly plastic behaviour implies unlimited ductility – such a material does not exist. Sufficient ductility must therefore be ensured in design.

In structural concrete, this can be achieved by the following conceptual measures:

→ neglect concrete tensile strength (for resistance) and

→ provide minimum reinforcement to avoid failure at crack formation

→ ensure yielding of reinforcement before concrete crushes



Theory of plasticity – Limit analysis

Generalised stresses and strains

The theorems of limit analysis equally apply in terms of generalised stresses and strains (deformations).

Generalised strains are obtained by introducing kinematic restrictions to the plastic strain increments $\dot{\boldsymbol{\varepsilon}}$. The generalised stress σ_j associated to a generalised strain $\dot{\varepsilon}_j$ results by integrating the stresses $\boldsymbol{\sigma}$ doing work on $\dot{\varepsilon}_j$.

Graphically, introducing kinematic restrictions corresponds to a projection of the yield surface *Y* to a lower-dimensional space *Z*, see top figure:

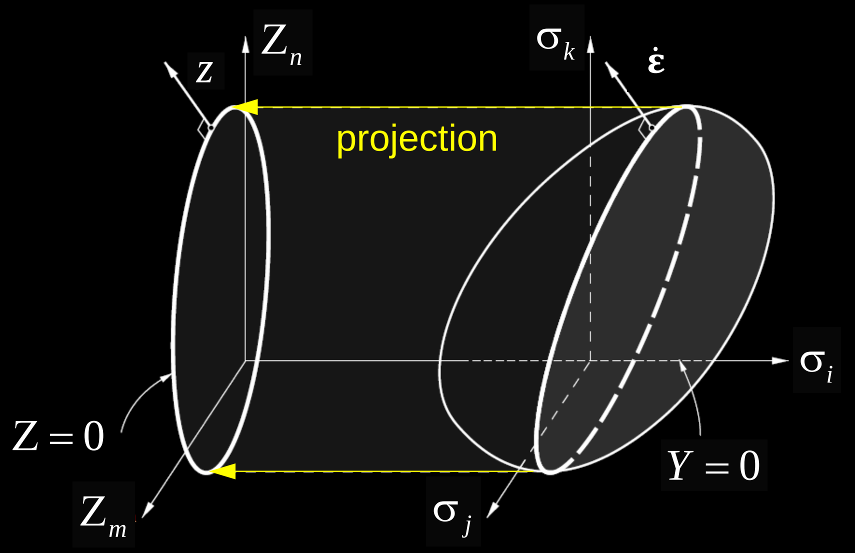
- projected values = *generalised stresses* {*Z*} *and strains* {*z*}
- stress components σ_i "lost" in the projection = *generalised reactions*, which assume the value maximising the ultimate load in terms of {*Z*}.

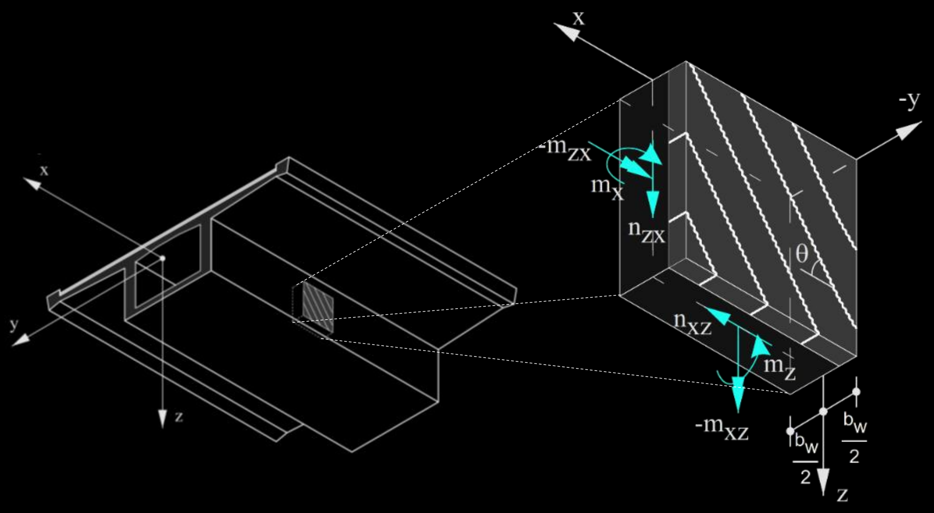
Example 1: Bernoulli's hypothesis (*M*-*N* interaction, see previous slide):

- axial strains $\dot{\varepsilon}_x$ → generalised strains $\{\dot{\chi}, \dot{\varepsilon}_m\}$
  axial stresses σ_x → generalised stresses {*M*, *N*} ("stress resultants")
  shear stresses τ_{zx} → generalised reaction {*V*}

Example 2 (bottom fig.): Shear + transverse bending, see Bridge Design:

- generalised stresses and strains: $\{n_z(=0), n_{zx}, m_z\}$ and $\{\dot{\varepsilon}_z, \dot{\gamma}_{zx}, \dot{\chi}_z\}$
  generalised reactions: $\{n_x, m_x, m_{zx}, v_{yx}, v_{yz}\}$





Theory of plasticity – Limit analysis

Main consequences of the theorems of limit analysis

- Residual stresses and restraints have no influence on the ultimate load (as long as the deformations remain small).
  (NB: in elastic solutions and particularly in stability problems, the failure load depends on residual stresses and restraints)
- Adding (subtracting) weightless material cannot decrease (increase) the ultimate load.
- Raising (lowering) the yield limit of the material in any region of a system cannot decrease (increase) its ultimate load.
- The ultimate load determined with a yield surface circumscribing (inscribing) the effective yield surface is an upper (lower) bound to the effective ultimate load.

Application of the theorems of limit analysis

The lower bound theorem of limits analysis is the most used in practice. Typical applications: strut-and-tie models and stress fields for membrane elements, the strip method for slabs.

Many national and international codes are based (in most cases only implicitly, and unknown to many people) on the lower bound theorem.

The upper limit theorem is particularly useful in assessing the structural safety of existing structures. An upper bound of the ultimate load can often be found with considerably less effort than required for the development of a statically admissible stress state that nowhere infringes the yield condition for given dimensions and reinforcement layout).

Pertinent assessment of structural safety

Pertinent assessment of structural safety

Solution:
maximise service life with
minimum interventions

- without excessive / unacceptable risk
- with limited resources

$\rightarrow$ pertinent assessment of structural safety is key

Structural safety		Reality: not ok	Reality: ok
Analysis	not ok	ok	resources
	ok	risk of collapse reputation	ok

Pertinent assessment of structural safety

- Analytically often much more demanding than the design of new structures
- High theoretical knowledge and experience required
- Verifications should only be carried out by qualified and experienced structural engineers.
- In practice, the complexity is often underestimated
  (similar analysis than for a new structure or by “reproducing" the structural verifications for construction)