1. Introduction

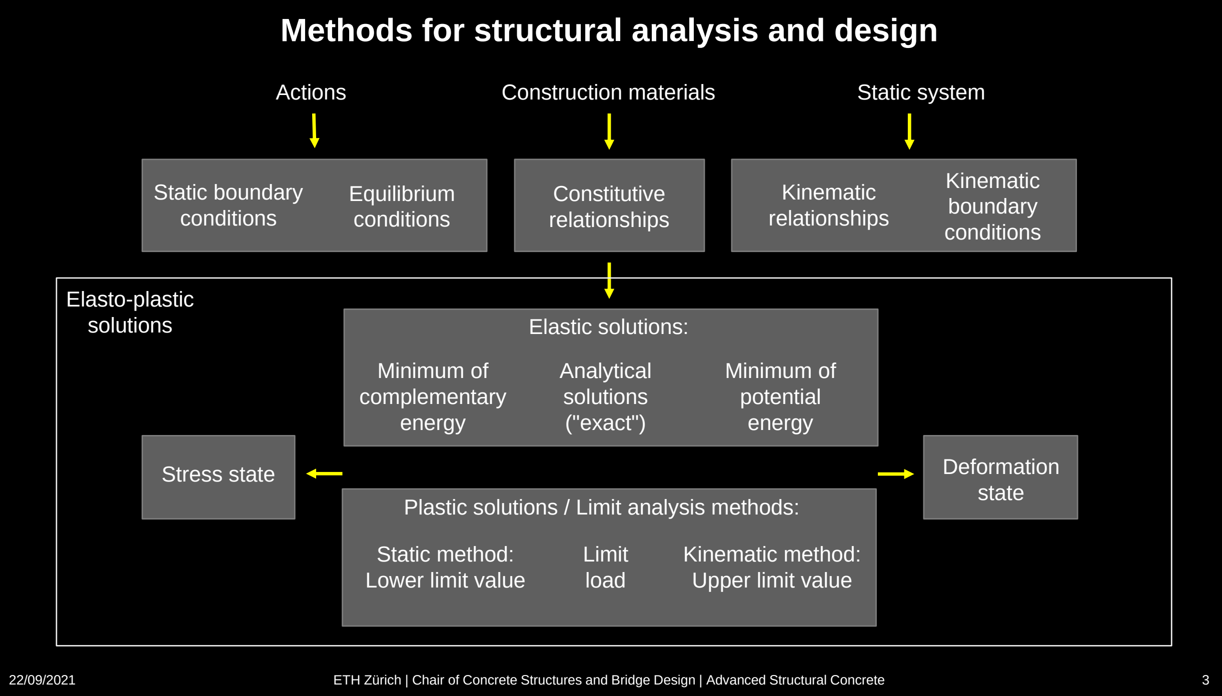
As an introduction, the most important principles of the limit analysis of the theory of plasticity are presented. This is largely a repetition from the lectures Stahlbeton I/II (identical slides). Students who have completed their bachelor's degree at ETH Zürich are already familiar with the limit analysis approaches (introduced in the course Baustatik II, used in Stahlbeton I/II, Stahlbau I/II, ...). The following short overview on this topic can serve as an introduction for the other students; however, it is advisable to deepen your knowledge through self-study.

The application of limit analysis is motivated in particular by the fact that, in contrast to elastic soulutions, it delivers clear results (see next page). This basic knowledge was stablished over 80 years ago (Melan 1938), but is unfortunately not well-known and often not given due attention. This brings about the risk of underestimating the importance of ductility provisions, especially for "new" construction materials without or with limited ductility (CFRP, GFRP and other fibre composites; steel fibre concrete and ultra high strength fibre concrete).

Learning objectives

Within this chapter, the students are able to:

- identify and distinguish the different methods to design/analyse concrete structures:
  - describe the differences between elastic solutions and plastic solutions (limit analysis methods).
  - differentiate between the approaches required to design a new structure and to assess an existing one.
  - explain the main assumptions and theorems of limit analysis and their consequences on structural design practice.



Methods for structural analysis and design

Actions

Construction materials

Static system

Static boundary conditions

Equilibrium conditions

Constitutive relationships

Kinematic relationships

Kinematic boundary conditions

Elasto-plastic solutions

Elastic solutions:

Minimum of complementary energy

Analytical solutions ("exact")

Minimum of potential energy

Stress state

Deformation state

Plastic solutions / Limit analysis methods:

Static method: Lower limit value

Limit load

Kinematic method: Upper limit value

The figure shows the different design methods for concrete structures. In many cases limit analysis methods (plastic solutions) are used, while often implicitly.

Comments on the figure:

- **Elastic solutions** require the consideration of equilibrium conditions, as well as constitutive and kinematic relationships. Elastic solutions implicitly assume that the loading history is exactly known and structural systems are free from residual stresses and restraints. However, this hardly ever applies to real concrete structures where the initial stresses, caused by restraint to imposed deformations (such as shrinkage strains), construction stages and other factors, are largely unknown. Therefore, design methods based on the comparison of elastically determined stresses to admissible stress values are generally not valid.

- **Plastic solutions (limit analysis methods)** solve the intrinsic problem of admissible stress design: If sufficient ductility is ensured (and no stability problems occur), the ultimate load is independent of residual stresses and restraints, as well as of the loading history. Therefore, the stress state (statically admissible stress field) and the deformation state (collapse mechanism) can be determined independently of each other. This usually simplifies the analysis considerably. However, plastic solutions do not provide any information about the serviceability behaviour of the structure nor about the deformation demand/capacity of the structure.

- **Elasto-plastic solutions** can also be implemented in numerical approaches. They allow formulating plastic solutions at Ultimate Limit State. Hence, share all the stated advantages stated for the plastic solutions. However, they also provide information about the deformation state, what allows verifying if the ductility of the structure is sufficient to develop the plastic (in classical plastic solutions the ductility is just assumed to be sufficient). The knowledge of the deformation state also allows estimating the behaviour of the structure under serviceability conditions (i.e. deflections, crack widths).

Dimensioning of new buildings

Ductile design

$x/d < 0.35$, conservative assumption for concrete compressive strength, adequate anchorage lengths, etc.

→ Prevent brittle failures

→ Ensure applicability of plastic design methods

Plastic dimensioning

→ Define desired load path and follow it consistently

→ Structural elements have clearly defined functions

Simple models sufficient

Restraint stresses negligible

Redundancy and robustness

Solving problems conceptually

Designing instead of calculating

Structural Concrete I/II
→ Bachelor's degree
→ Specialisation ... (≠ Construction)

The theory of plasticity simplifies the dimensioning of new buildings. Principle of the design of new structures:

"The engineer determines how the structure carries the loads and designs it accordingly (dimensions, reinforcement)".

(and not: "The structure tells the engineer what to calculate")

Theory of plasticity – Limit analysis

Principle of maximum dissipation energy

Equivalent to: convexity of the yield condition + orthogonality of the plastic strain increments to the yield surface
→ Maximum dissipation energy (= basis of the limit analysis of the theory of plasticity)

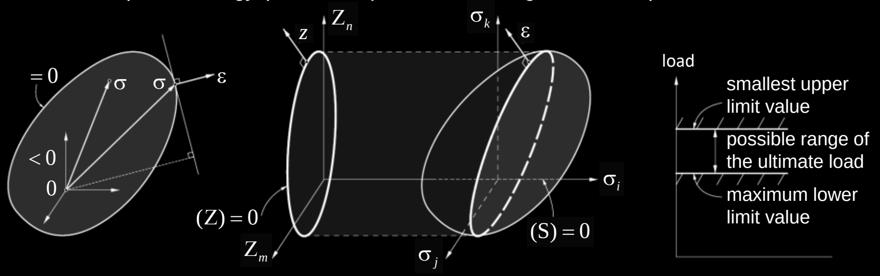
Generalised stresses and deformations

Introduction of kinematic restrictions ↔ Projections of the yield surface (Example: Hypothesis of Bernoulli for bending:
$\boldsymbol{\sigma} \rightarrow \{M, N\}$ and $\dot{\boldsymbol{\varepsilon}} \rightarrow \{\dot{\chi}, \dot{\varepsilon}_0\}$)

Projected values = *generalised stresses and deformations*

Stress components “lost” in the projection = *generalised reactions*

The principle of maximum dissipation energy (and others) is also valid in generalised quantities



Additional remarks to the figure:

A generalised reaction $r = \sigma_i$ can take arbitrary values; this corresponds to a projection of the yield surface into any plane σ_i = const. (for example $\sigma_i = 0$).

The principle of maximum dissipation energy requires the yield surface to be convex and the strain increments to be orthogonal to the yield surface (a two-dimensional graphical example could be used to verify that the dissipation work is not maximum for concave areas of the yield condition or non-orthogonal strain increments).

On the other hand, convexity and orthogonality follow when maximum dissipation energy is given.

Theory of plasticity – Limit analysis

Lower bound (static) theorem

Every loading for which it is possible to specify a statically admissible stress state that does not infringe the yield condition is not greater than the ultimate load.

(statically admissible: a stress state satisfying equilibrium and static boundary conditions)

Upper bound (kinematic) theorem

Every loading that results from equating the work of external forces for a kinematically admissible deformation state with the associated dissipation work is not less than the limit load.

(kinematically admissible: kinematic relationships and kinematic boundary conditions are fulfilled)

Compatibility theorem

A load for which a complete solution can be specified is equal to the ultimate load.

(complete solution: statically admissible stress state that does not infringe the yield condition and a compatible kinematically admissible state of deformation can be specified for that load.)

Additional remarks to the figure:

The lower bound theorem of limit analysis guarantees a safe design, provided the applicability conditions are fulfilled (i.e. sufficient deformation capacity). Most design methods in structural concrete are therefore based on the lower bound theorem of the theory of plasticity. The theory of plasticity ensures the safe application of several design methods, for example:

... strut-and-tie models and stress fields
... equilibrium solutions for slabs (e.g. strip method)
... yield conditions for membrane elements and slabs
... etc.

Finding a complete solution is difficult (if not impossible, unless numerical approaches are used), but the advantage of the compatibility theorem is that it does not require performing neither a detailed compatibility check nor a plasticity verification for the mechanism found.

Theory of plasticity – Limit analysis

Main consequences of the theorems of limit analysis

- Residual stresses and restraints have no influence on the ultimate load (as long as the resulting deformations remain infinitesimally small).
  (NB: This applies only to limit analysis methods; in elastic solutions and particularly in stability problems, the failure load depends on residual stresses and restrains)
- Adding (subtracting) weightless material cannot decrease (increase) the ultimate load.
- Raising (lowering) the yield limit of the material in any region of a system cannot decrease (increase) its ultimate load.
- The ultimate load that can be calculated with a yield surface circumscribing (inscribing) the effective yield surface forms an upper (lower) bound to the effective ultimate load.

Application of the theorems of limit analysis

The lower bound theorem of limits analysis is the most used in practice. Typical applications: strut-and-tie models and stress fields for membrane elements, the strip method for slabs.

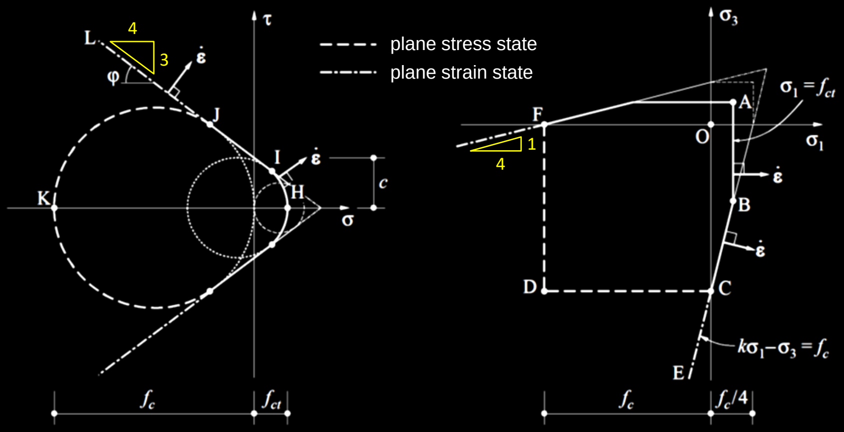
Many national and international codes are based (in most cases only implicitly, and unknown to many people) on the lower bound theorem.

In practice, the upper limit theorem is particularly helpful in assessing the structural safety of existing structures. (Allows limiting the ultimate load. This is often possible with considerably less effort than the development of a statically admissible stress state that does not violate the yield condition anywhere.)

Theory of plasticity – Limit analysis

Concrete - Modified Coulomb yield surface

Normal concrete: $\tan(\varphi) = 0.75 \rightarrow c = f_c/4$, φ = approx. 37°



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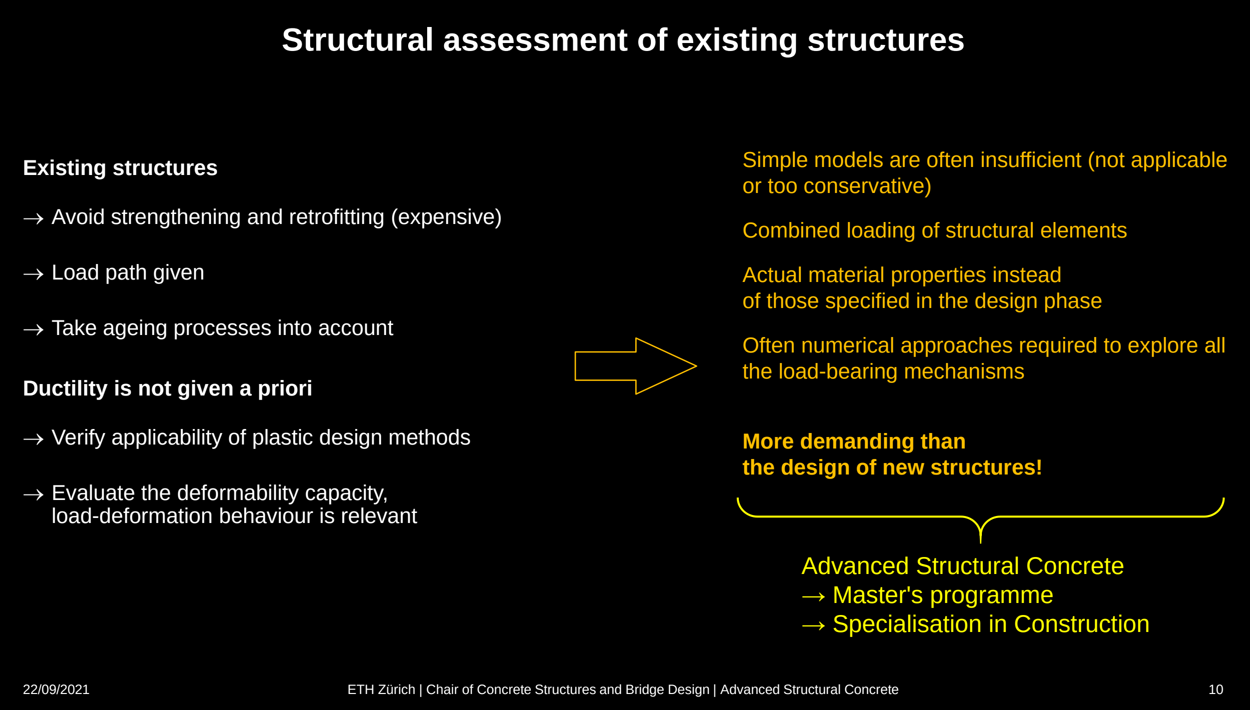

Structural assessment of existing structures

Existing structures

→ Avoid strengthening and retrofitting (expensive)

→ Load path given

→ Take ageing processes into account

Ductility is not given a priori

→ Verify applicability of plastic design methods

→ Evaluate the deformability capacity,
load-deformation behaviour is relevant

Simple models are often insufficient (not applicable or too conservative)

Combined loading of structural elements

Actual material properties instead
of those specified in the design phase

Often numerical approaches required to explore all the load-bearing mechanisms

**More demanding than
the design of new structures!**

Advanced Structural Concrete
→ Master's programme
→ Specialisation in Construction

In the case of existing structures, the problems are more complex and usually cannot be avoided conceptually.

The structure was designed and dimensioned by someone else, often on the basis of models that are not reliable from today's point of view. In particular, the prerequisites for the application of plastic design and verification methods are often not met.

Unfortunately, many engineers and clients are not yet fully aware of the complexity of the structural safety inspection of existing structures and the associated responsibility. The misbelief that any engineer can check the structural safety («the building has been standing for 30 years») is unfortunately not uncommon.

The structural assessment of existing structures is often extremely complex:

- An excessively conservative assessment leads to unnecessary and expensive strengthening (the less bad alternative).
- Over-predicting the capacity of the structure and underestimating problems is very risky both for the engineer and for the client.