Derivation of the normal moment yield condition for orthorgonal reinforcement

Bending load in arbitrary direction n (Angle ϕ measured from x-Direction)

$$m_n(\phi) = m_x \cdot \cos(\phi)^2 + m_y \cdot \sin(\phi)^2 + m_{xy} \cdot \sin(2\cdot\phi)$$

Bending resistance in arbitrary direction n

$$m_{nu}(\phi) = m_{xu} \cdot \cos(\phi)^2 + m_{yu} \cdot \sin(\phi)^2$$

Condition (must be valid for all ϕ), insert upper equations

$$m_{nu}(\phi) \ge m_n(\phi) \qquad (m_{xu} - m_x)\cdot\cos(\phi)^2 + (m_{yu} - m_y)\cdot\sin(\phi)^2 - m_{xy}\cdot\sin(2\cdot\phi) \ge 0$$

Controlling direction (Minimum) Variant 1

$$\frac{d}{d\phi}m_{nu} - \frac{d}{d\phi}m_n = 0 \qquad \frac{d}{d\phi}\left[(m_{xu} - m_x)\cdot\cos(\phi)^2 + (m_{yu} - m_y)\cdot\sin(\phi)^2 - m_{xy}\cdot\sin(2\cdot\phi)\right] = 0$$

$$-2\cdot\cos(\phi)\cdot\sin(\phi)\cdot(m_{xu} - m_x) - 2\cdot m_{xy}\cdot\cos(2\cdot\phi) + 2\cdot\cos(\phi)\cdot\sin(\phi)\cdot(m_{yu} - m_y) = 0$$

Solve for (m.xu-m.x) resp. (m.yu-m.y)

$$2\cdot\cos(\phi)\cdot\sin(\phi)\cdot\left[(m_{yu} - m_y) - (m_{xu} - m_x)\right] = 2\cdot m_{xy}\cdot\cos(2\cdot\phi)$$

$$(m_{yu} - m_y) - (m_{xu} - m_x) = 2\cdot m_{xy}\cdot\cot(2\cdot\phi)$$

$$(m_{yu} - m_y) = (m_{xu} - m_x) + 2\cdot m_{xy}\cdot\cot(2\cdot\phi)$$

$$(m_{xu} - m_y) = (m_{yu} - m_y) - 2\cdot m_{xy}\cdot\cot(2\cdot\phi)$$

Insert into condition m.nu = m.n (m.yu-m.y substituted), simplified

$$(m_{xu} - m_x)\cdot\cos(\phi)^2 + (m_{yu} - m_y)\cdot\sin(\phi)^2 - m_{xy}\cdot\sin(2\cdot\phi)$$

$$(m_{xu} - m_x)\cdot\cos(\phi)^2 + \left[(m_{xu} - m_x) + 2\cdot m_{xy}\cdot\cot(2\cdot\phi)\right]\cdot\sin(\phi)^2 - m_{xy}\cdot\sin(2\cdot\phi)$$

$$m_{xu} - m_x = m_{xy}\cdot\left(\sin(2\cdot\phi) - 2\cdot\cot(2\cdot\phi)\cdot\sin(\phi)^2\right) = \tan(\phi)\cdot m_{xy}\cdot\left(2\cdot\cos(\phi)^2 - \frac{\cos(\phi)^2 - \sin(\phi)^2}{\sin(\phi)}\cdot\sin(\phi)\right)$$

$$m_{xu} - m_x = \tan(\phi)\cdot m_{xy}\cdot\left[2\cdot\cos(\phi)^2 - \left[\cos(\phi)^2 - \left(1 - \cos(\phi)^2\right)\right]\right]$$

$$m_{xu} - m_x - m_{xy}\cdot\tan(\phi) \qquad m_{xu} - m_x = m_{xy}\cdot\tan(\phi)$$

Insert into condition m.nu = m.n (m.x-m.x substituted), simplified

$$(m_{xu} - m_x)\cdot\cos(\phi)^2 + (m_{yu} - m_y)\cdot\sin(\phi)^2 - m_{xy}\cdot\sin(2\cdot\phi)$$

$$\left[(m_{yu} - m_y) - 2\cdot m_{xy}\cdot\cot(2\cdot\phi)\right]\cdot\cos(\phi)^2 + (m_{yu} - m_y)\cdot\sin(\phi)^2 - m_{xy}\cdot\sin(2\cdot\phi)$$

$$m_{yu} - m_y - m_{xy}\cdot\cot(\phi) \qquad m_{yu} - m_y = m_{xy}\cdot\cot(\phi)$$

Multiplication of the two equations

$$\left(m_{xu} - m_x\right)\cdot\left(m_{yu} - m_y\right) = m_{xy}^2 \qquad \tan(\phi) = \frac{\left(m_{xu} - m_x\right)}{m_{xy}} = \frac{m_{xy}}{m_{yu} - m_y} = \sqrt{\frac{m_{xu} - m_x}{m_{yu} - m_y}}$$

Controlling direction (Minimum) Variant 2

$$\frac{d}{d\phi}m_{nu} = \frac{d}{d\phi}m_n \qquad \frac{d}{d\phi}m_{nu} \quad 2\cdot m_{yu}\cdot\cos(\phi)\cdot\sin(\phi) - 2\cdot m_{xu}\cdot\cos(\phi)\cdot\sin(\phi)$$

$$\frac{d}{d\phi}m_n \quad 2\cdot m_{xy}\cdot\cos(2\cdot\phi) - 2\cdot m_x\cdot\cos(\phi)\cdot\sin(\phi) + 2\cdot m_y\cdot\cos(\phi)\cdot\sin(\phi)$$

$$2\cdot m_{xy}\cdot\cos(2\cdot\phi) - 2\cdot m_x\cdot\cos(\phi)\cdot\sin(\phi) + 2\cdot m_y\cdot\cos(\phi)\cdot\sin(\phi) = 2\cdot m_{yu}\cdot\cos(\phi)\cdot\sin(\phi) - 2\cdot m_{xu}\cdot\cos(\phi)\cdot\sin(\phi)$$

$$\frac{2\cdot m_{xy}\cdot\cos(2\cdot\phi)}{2\cdot\cos(\phi)\cdot\sin(\phi)} - m_x + m_y = m_{yu} - m_{xu}$$

$$-\left(m_{xu} - m_x\right) + \left(m_{yu} - m_y\right) - 2\cdot m_{xy}\cdot\cot(2\cdot\phi) = 0 \qquad (1)$$

$$\left(m_{xu} - m_x\right)\cdot\cos(\phi)^2 + \left(m_{yu} - m_y\right)\cdot\sin(\phi)^2 - m_{xy}\cdot\sin(2\cdot\phi) \ge 0 \qquad (2)$$

$$\left(m_{xu} - m_x\right)\cdot\cos(\phi)^2 + \left(m_{yu} - m_y\right)\cdot\sin(\phi)^2 - m_{xy}\cdot(2\cdot\cos(\phi)\cdot\sin(\phi)) \ge 0$$

$$\left(m_{xu} - m_x\right)\cdot\cot(\phi)^2 + \left(m_{yu} - m_y\right) - m_{xy}\cdot 2\cdot\cot(\phi) \qquad (2)\,/\ \sin(\phi)^2$$

$$\left(m_{xu} - m_x\right)\cdot\left(1 + \cot(\phi)^2\right) - m_{xy}\cdot 2\cdot\cot(\phi) = -2\cdot m_{xy}\cdot\cot(2\cdot\phi) \qquad (2)\,/\ \sin(\phi)^2 - (1)$$

$$\left(m_{xu} - m_x\right)\cdot\left(1 + \cot(\phi)^2\right) - m_{xy}\cdot 2\cdot\cot(\phi) = -m_{xy}\cdot\frac{\cot(\phi)^2 - 1}{\cot(\phi)} = -m_{xy}\cdot\cot(\phi) + m_{xy}\cdot\tan(\phi)$$

$$\left(m_{xu} - m_x\right)\cdot\left(1 + \cot(\phi)^2\right) - m_{xy}\cdot 2\cdot\cot(\phi) = -m_{xy}\cdot\frac{\cot(\phi)^2 - 1}{\cot(\phi)} = -m_{xy}\cdot\cot(\phi) + m_{xy}\cdot\tan(\phi)$$

$$\left(m_{xu} - m_x\right)\cdot\left(1 + \cot(\phi)^2\right) = m_{xy}\cdot(\cot(\phi) + \tan(\phi)) = m_{xy}\cdot\tan(\phi)\cdot\left(1 + \cot(\phi)^2\right)$$

$$\left(m_{xu} - m_x\right) = m_{xy}\cdot\tan(\phi) \qquad m_{xu} \ge m_x + m_{xy}\cdot\tan(\phi)$$

$$\left(m_{xu} - m_x\right) + \left(m_{yu} - m_y\right)\cdot\tan(\phi)^2 - m_{xy}\cdot 2\cdot\tan(\phi) \qquad (2)\,/\ \cos(\phi)^2$$

$$\left(m_{yu} - m_y\right)\cdot\left(1 + \tan(\phi)^2\right) - m_{xy}\cdot 2\cdot\tan(\phi) = 2\cdot m_{xy}\cdot\cot(2\cdot\phi) \qquad (2)\,/\ \cos(\phi)^2 + (1)$$

$$\left(m_{yu} - m_y\right)\cdot\left(1 + \tan(\phi)^2\right) - m_{xy}\cdot 2\cdot\tan(\phi) = m_{xy}\cdot\frac{\cot(\phi)^2 - 1}{\cot(\phi)} = m_{xy}\cdot\cot(\phi) - m_{xy}\cdot\tan(\phi)$$

$$\left(m_{yu} - m_y\right)\cdot\left(1 + \tan(\phi)^2\right) = m_{xy}\cdot(\cot(\phi) + \tan(\phi)) = m_{xy}\cdot\cot(\phi)\cdot\left(1 + \tan(\phi)^2\right)$$

$$\left(m_{yu} - m_y\right) = m_{xy}\cdot\cot(\phi) \qquad m_{yu} \ge m_y + m_{xy}\cdot\cot(\phi)$$

$$\left(m_{xu} - m_x\right)\cdot\left(m_{yu} - m_y\right) = m_{xy}^2 \qquad \tan(\phi) = \frac{\left(m_{xu} - m_x\right)}{m_{xy}} = \frac{m_{xy}}{m_{yu} - m_y} = \sqrt{\frac{m_{xu} - m_x}{m_{yu} - m_y}}$$