1. Introduction

As an introduction, the most important principles of the limit analysis of the theory of plasticity are presented. This is largely a repetition from the lectures Stahlbeton I/II (identical slides). Students who have completed their bachelor's degree at ETH Zürich are already familiar with the limit analysis approaches (introduced in the course Baustatik II, used in Stahlbeton I/II, Stahlbau I/II, ...). The following short overview on this topic can serve as an introduction for the other students; however, it is advisable to deepen your knowledge through self-study.

The application of limit analysis is motivated in particular by the fact that, in contrast to elastic solutions, it delivers clear results (see next page). This basic knowledge was established over 80 years ago (Melan 1938), but is unfortunately not well-known and often not given due attention. This brings about the risk of underestimating the importance of ductility provisions, especially for "new" construction materials without or with limited ductility (CFRP, GFRP and other fibre composites; steel fibre concrete and ultra high strength fibre concrete).

Learning objectives

Based on this chapter, the students are able to:

- understand the relevance of deepening their knowledge of the load-bearing behaviour of concrete structures to
  - conceive and design efficient new structures hence reduce greenhouse gas emissions
  - pertinently assess existing structures hence extend their lifespan with minimum intervention

- identify and distinguish different methods to *analyse and design* concrete structures:
  - describe the differences between elastic solutions and plastic solutions (limit analysis methods).
  - differentiate between approaches to design a new structure and to assess an existing one.
  - explain the theorems of limit analysis, the underlying assumptions and their consequences on structural design practice.

Why Advanced Structural Concrete?

Impact of structural engineer in a design office

Source: The Institution of Structural Engineers – How to calculate embodied carbon

Holistic environmental sustainability

$$\text{annual emissions per use} = \frac{\dfrac{\text{emissions}}{\text{mass}} \cdot \dfrac{\text{mass}}{\text{floor area}} \cdot \dfrac{\text{floor area}}{\text{use}}}{\text{service life}}$$

- annual emissions: drastic reduction required
- use: societal needs / demands
- emissions / mass: reduce emissions of materials
- mass / floor area: design efficient structures
- floor area / use: reduce comfort level
- service life: ensure durability / extend lifespan

[Gebhard 2023, based on Flatt and Wangler 2022]

Holistic environmental sustainability

requires profound knowledge beyond the fundamentals taught in BSc courses

design efficient structures

$$annual\ emissions\ per\ use = \frac{\dfrac{emissions}{mass} \cdot \boxed{\dfrac{mass}{floor\ area}} \cdot \dfrac{floor\ area}{use}}{\boxed{service\ life}}$$

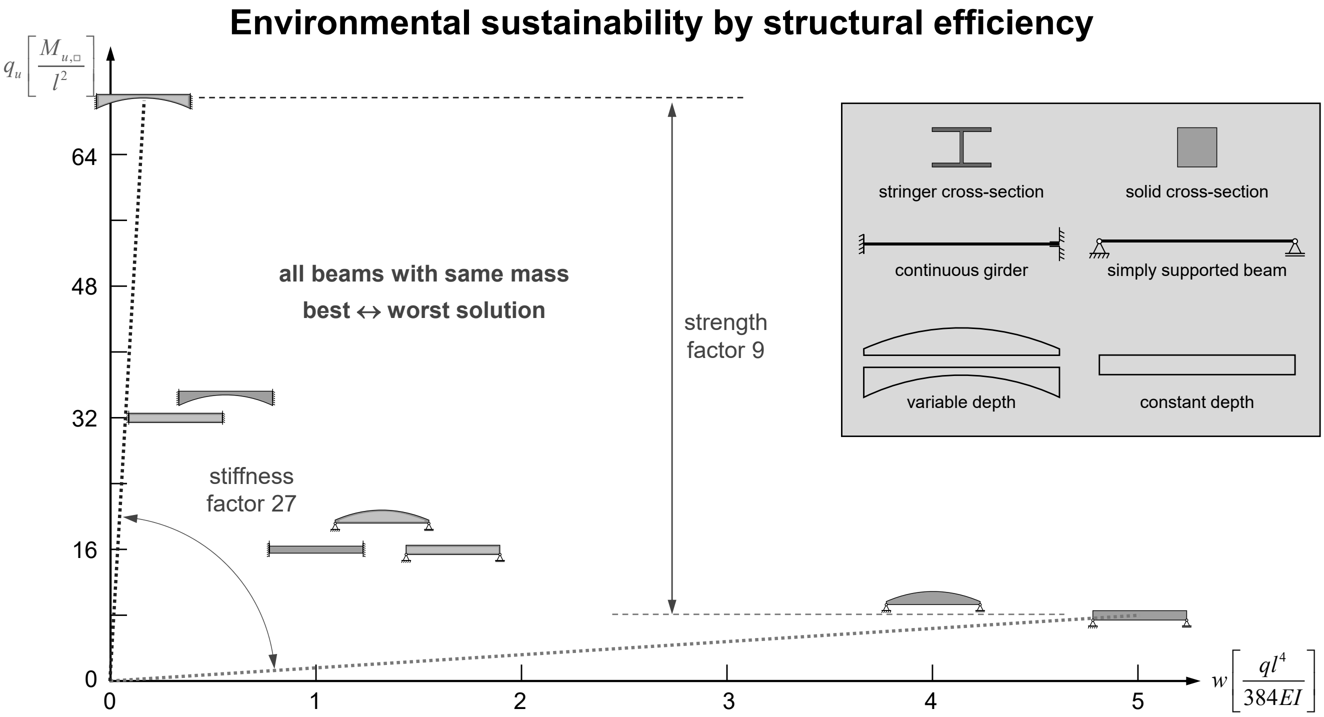
ensure durability / extend lifespan without interventions

[Gebhard 2023, based on Flatt and Wangler 2022]

Environmental sustainability by structural efficiency

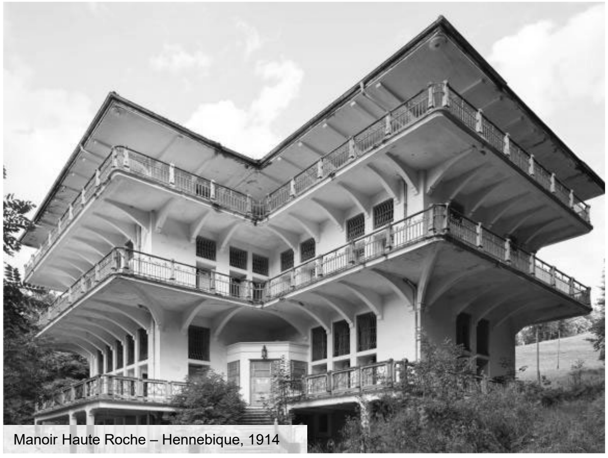
Gatti Wool Factory – Pier Luigi Nervi, 1951

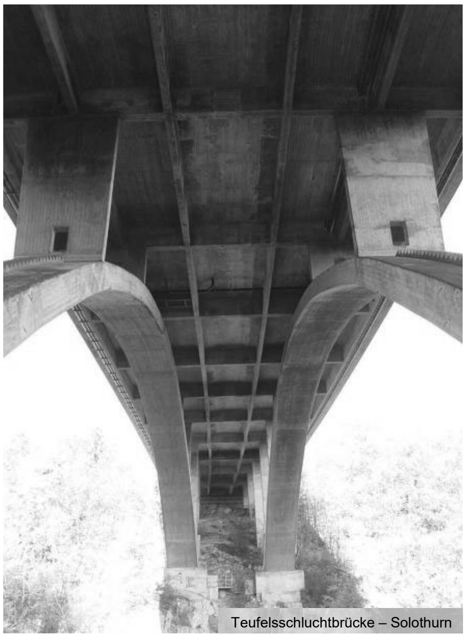
Ausgleichsbecken Les Marécottes - A. Sarrasin - 1926



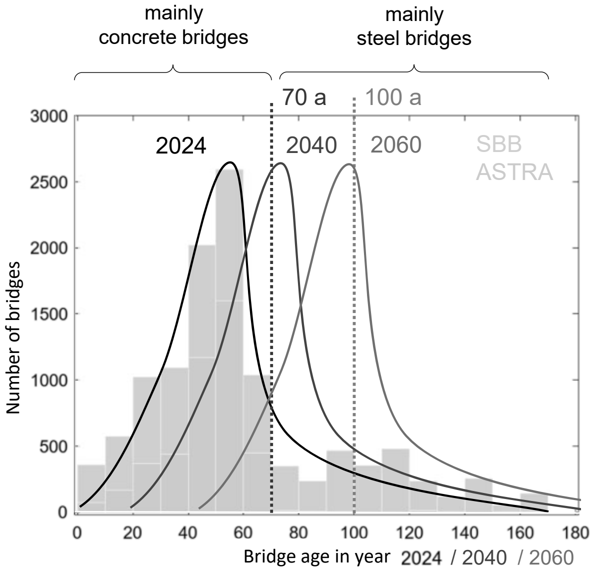
Environmental sustainability by structural efficiency
$q_u \left[\frac{M_{u,\square}}{l^2}\right]$
64
48
32
16
0
all beams with same mass
best ↔ worst solution
strength
factor 9
stiffness
factor 27
stringer cross-section
solid cross-section
continuous girder
simply supported beam
variable depth
constant depth
0
1
2
3
4
5
$w \left[\frac{ql^4}{384EI}\right]$

Environmental sustainability by structural efficiency


Manoir Haute Roche – Hennebique, 1914

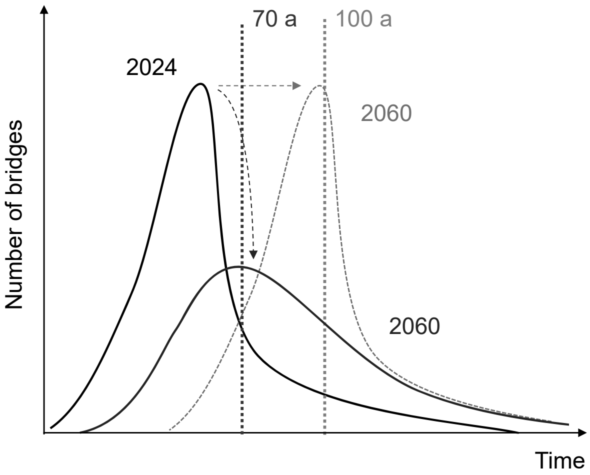

Teufelsschluchtbrücke – Solothurn

Stretching the lifespan of existing (concrete) structures



Many bridges are reaching their planned service life
Massive number of existing structures
(CH: 40’000, Europe 1 Mio, USA 650’000, …)

Stretching the lifespan of existing (concrete) structures



Solution:
maximise service life with
minimum interventions

- without excessive / unacceptable risk
- with limited resources

→ pertinent assessment of structural safety is key

Structural safety		**Reality**	
		not ok	ok
Analysis	not ok	ok	resources
	ok	risk of collapse reputation	ok

Dimensioning of new structures

Ensure ductility by conceptual measures

→ neglect concrete tensile strength and provide minimum reinforcement to avoid brittle failure at crack formation

→ ensure failure by yielding of reinforcement
… limit reinforcement ratio (e.g. $x/d < 0.35$ in bending)
… use conservative value of concrete compressive strength

→ verify deformation capacity (rarely in new design)

Design based on lower-bound theorem of plasticity theory

→ define and consistently follow design load path

→ structural elements have clearly defined functions (e.g. web with pure in-plane loading)

→ conservative design

Simple models sufficient

Restraint stresses can be neglected

Redundancy and robustness

Solving problems conceptually

Designing instead of calculating

Structural Concrete I/II

BSc degree

"The engineer tells the structure how to carry the loads"

~~"The structure tells the engineer what to calculate"~~

The theory of plasticity simplifies the dimensioning of new buildings. Principle of the design of new structures:

"The engineer determines how the structure carries the loads and designs it accordingly (dimensions, reinforcement)".

(and not: "The structure tells the engineer what to calculate")

Structural assessment of existing structures

Maximise service life with minimum interventions

→ avoid strengthening and retrofitting

→ without excessive / unacceptable risk

Challenges

→ dimensions and reinforcement are given, cannot be designed for a chosen load path

→ effect of deterioration (corrosion, ASR, …)?

→ ductility often not given a priori (lacking minimum reinforcement, heavy prestressing with $x/d > 0.35$, …)

→ structural elements with several functions (e.g. web with combined in-plane and out-of-plane loading)

→ Evaluate the deformation capacity, load-deformation behaviour is relevant

⇒

Simple models often insufficient (not applicable or overly conservative)

Use of updated material properties

Verification of deformation capacity required

Numerical approaches required to exploit load-bearing capacity

More demanding than design of new structures!

Advanced Structural Concrete

MSc, Major in Structural Engineering

In the case of existing structures, the problems are more complex and usually cannot be avoided conceptually.

The structure was designed and dimensioned by someone else, often on the basis of models that are not reliable from today's point of view. In particular, the prerequisites for the application of plastic design and verification methods are often not met.

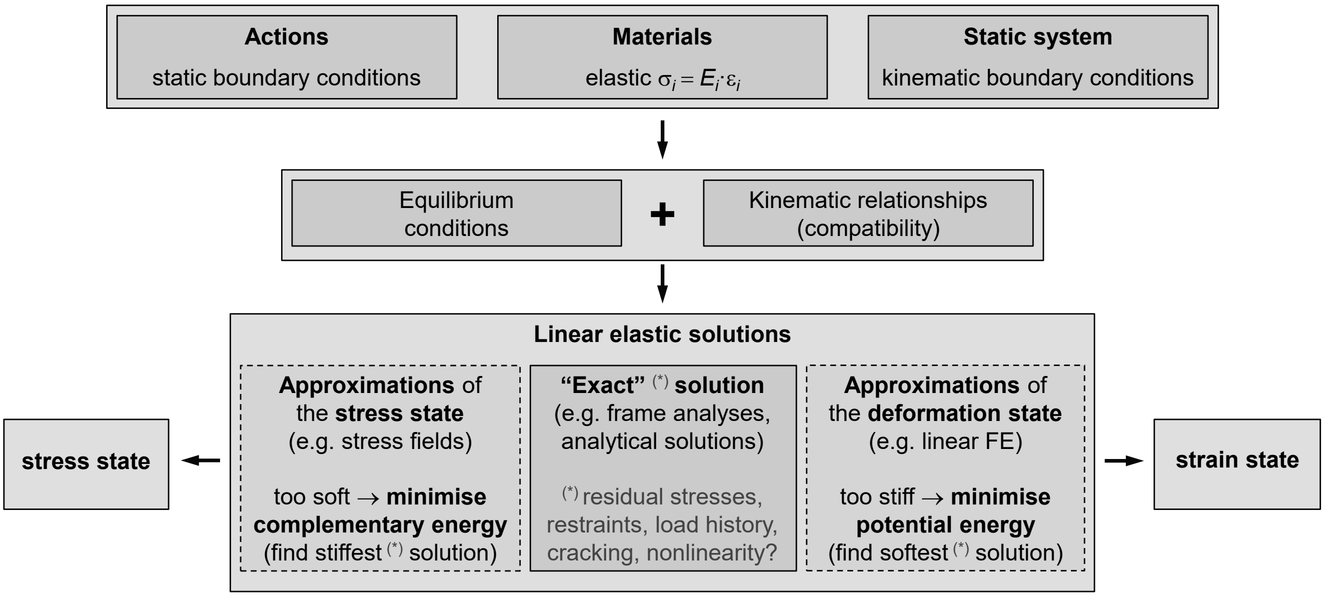
Unfortunately, many engineers and clients are not yet fully aware of the complexity of the structural safety inspection of existing structures and the associated responsibility. The misbelief that any engineer can check the structural safety («the building has been standing for 30 years») is unfortunately not uncommon.

The structural assessment of existing structures is often extremely complex:

- An excessively conservative assessment leads to unnecessary and expensive strengthening (the less bad alternative).
- Over-predicting the capacity of the structure and underestimating problems is very risky both for the engineer and for the client.

Methods of analysis and design

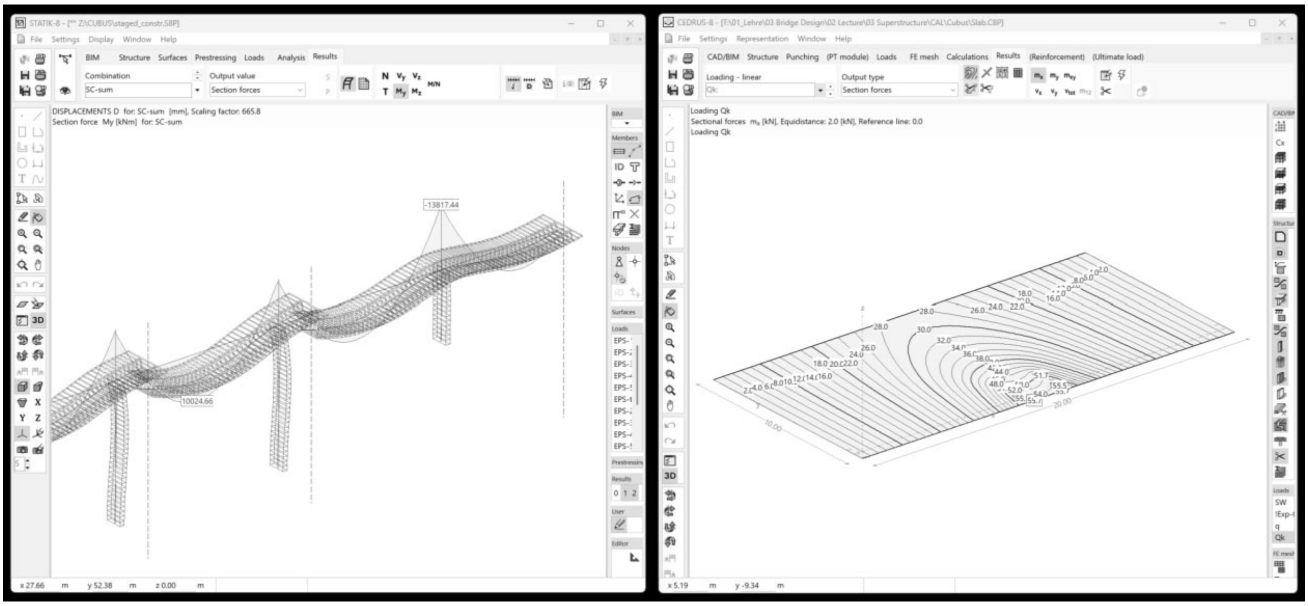
Structural analysis and design – Linear elastic (FE) analyses



This and the following slides schematically show the different analysis and design methods for reinforced concrete (RC) structures. In many cases limit analysis methods (plastic solutions) are used, while often implicitly.

Elastic solutions assuming homogeneous isotropic behaviour require the consideration of equilibrium conditions as well as constitutive and kinematic relationships (compatibility). They yield both stress as well as strain state (deformations)., but if the concrete cracks, the results are not realistic. Furthermore, elastic solutions implicitly assume that the loading history is known and structural systems are free from residual stresses and restraints. However, this hardly ever applies to real concrete structures; on the contrary, initial stresses e.g. caused by restrained shrinkage are largely unknown. Hence, "admissible stress design" (comparing elastically determined stresses to admissible stress values), commonly used until the 1960s, is generally invalid and has been superseded by ultimate limit state design.

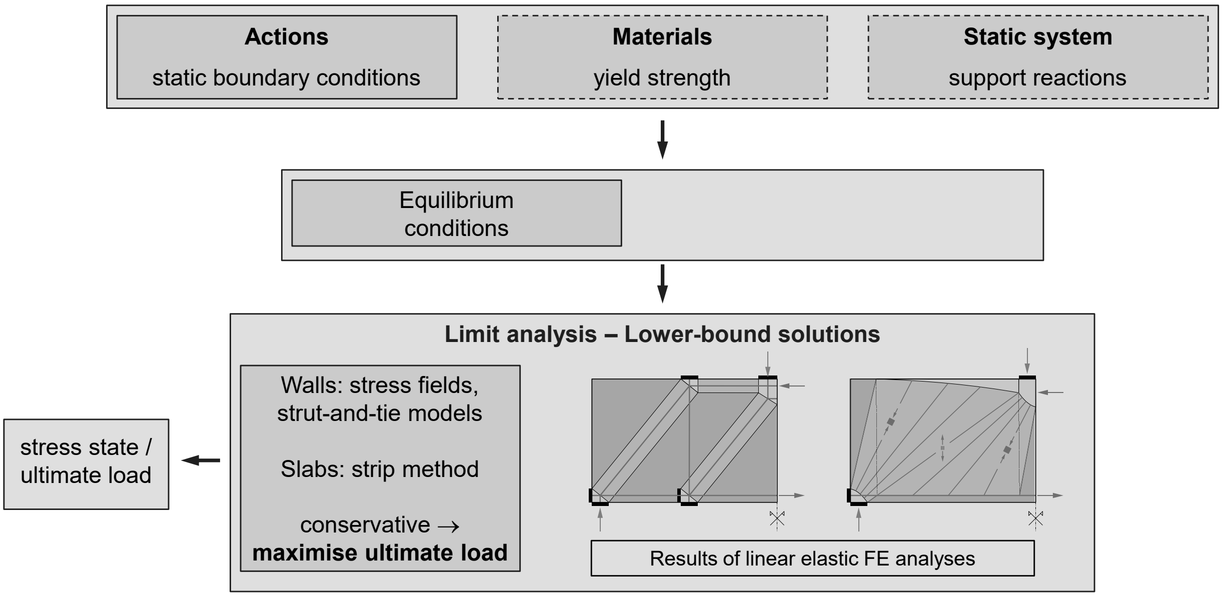
Structural analysis and design – Linear elastic (FE) analyses



[STATIK / CEDRUS software (CUBUS AG, Zürich)]

Examples of linear FE analyses. Note that frame analyses are exact (within the assumptions of Navier-Bernoulli beam theory), while plate and shell analyses are approximations, inherently too stiff as the finite elements used restrict kinematics.

Structural analysis and design – Limit analysis methods (i)



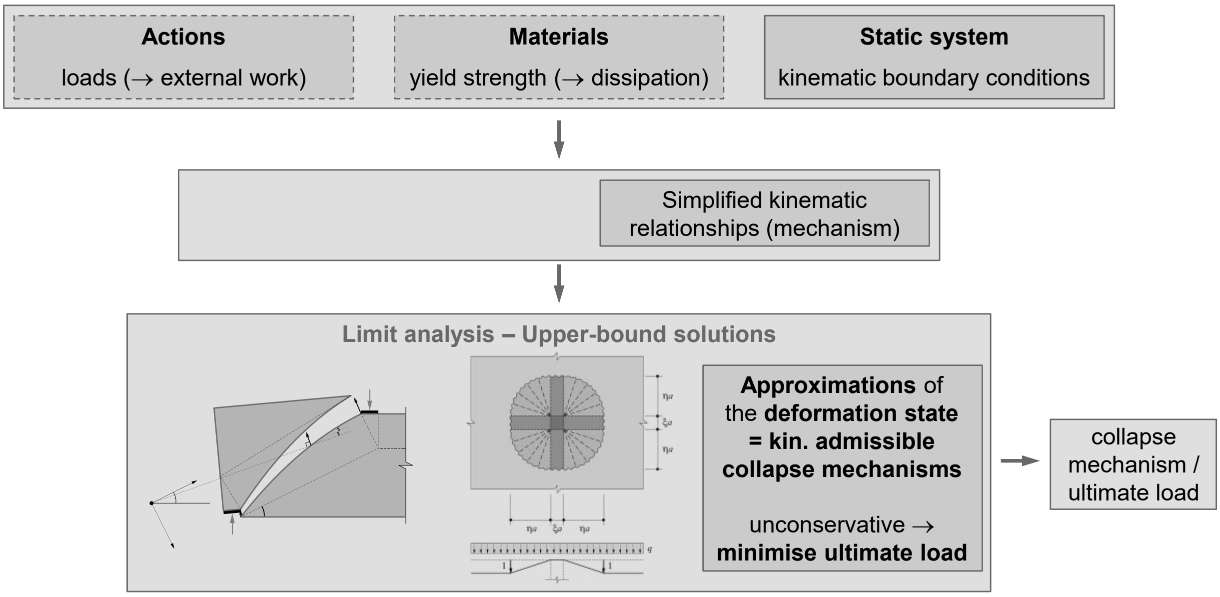
Limit analysis methods solve the intrinsic problem of admissible stress design: If sufficient ductility is ensured (and no stability problems occur), the ultimate load is independent of residual stresses and restraints, as well as of the loading history.

The stress state (statically admissible stress field, this slide) and the deformation state (collapse mechanism, next slide) can be determined independently, which simplifies the analysis considerably.

However, limit analysis does not provide any information about the serviceability behaviour of the structure nor about the deformation demand / capacity of the structure.

Due to the unknown initial stresses (residual stresses, restraints), nor realistically account for stiffnesses (cracking), the results of linear elastic finite element analyses are not exact (see previous slide). However, they can be considered as one (of many) lower-bound solution, since equilibrium and static boundary conditions are fulfilled.

Structural analysis and design – Limit analysis methods (ii)



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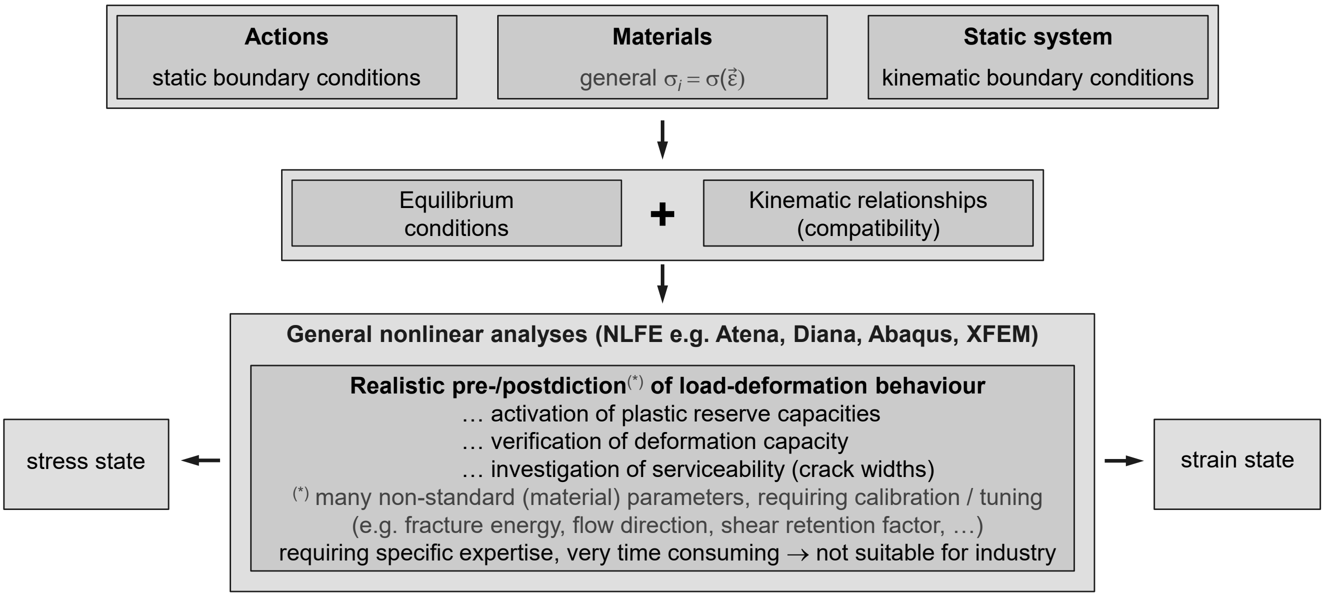
Structural analysis and design – General NLFE analyses

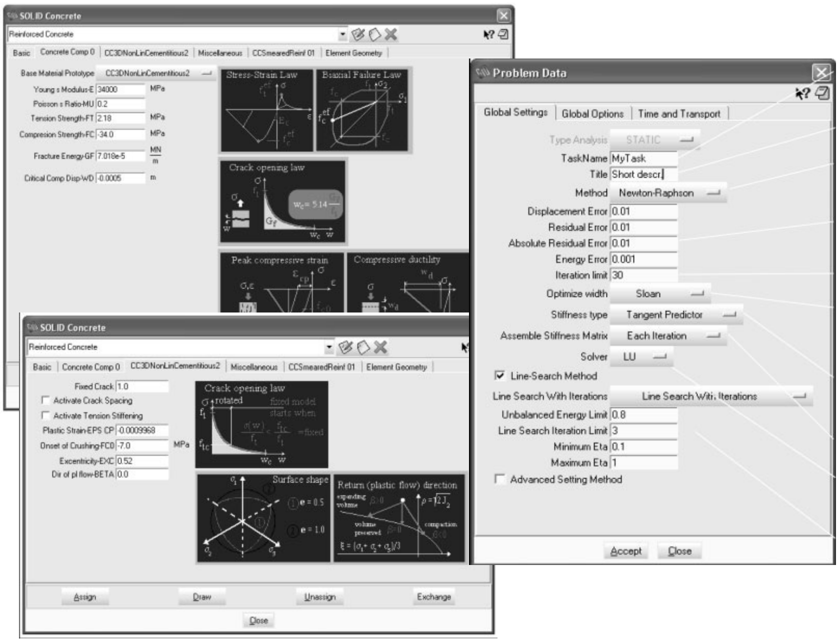


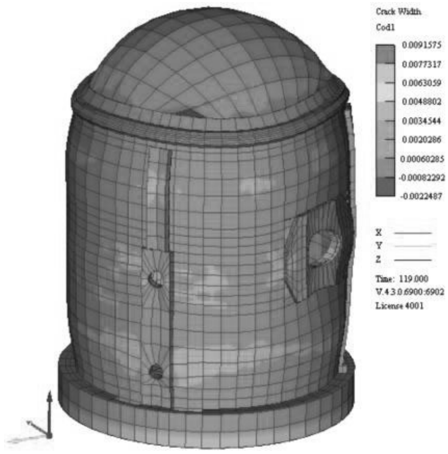
General nonlinear finite element analyses are usually implemented in numerical approaches. If pertinently modelled, they combine the advantages of elastic analysis and limit analysis methods, as they provide information about the stress and strain state and allow the deformation demand / capacity to be verified as well as the serviceability behaviour (i.e. deflections, crack widths) to be checked.

However, **general NLFE analyses** (e.g. Atena, Diana, Abaqus) are not suitable for use in industry, as (i) many non-standard (material) parameters, such as fracture energy, need to be calibrated and models tuned to obtain realistic load-deformation pre- or rather post-dictions, (ii) they require specific expertise and (iii) are very time consuming.

General NLFE analyses are therefore mainly used in academia, e.g. for parametric studies calibrated on experimental data of a basic case.

Structural analysis and design – General NLFE analyses

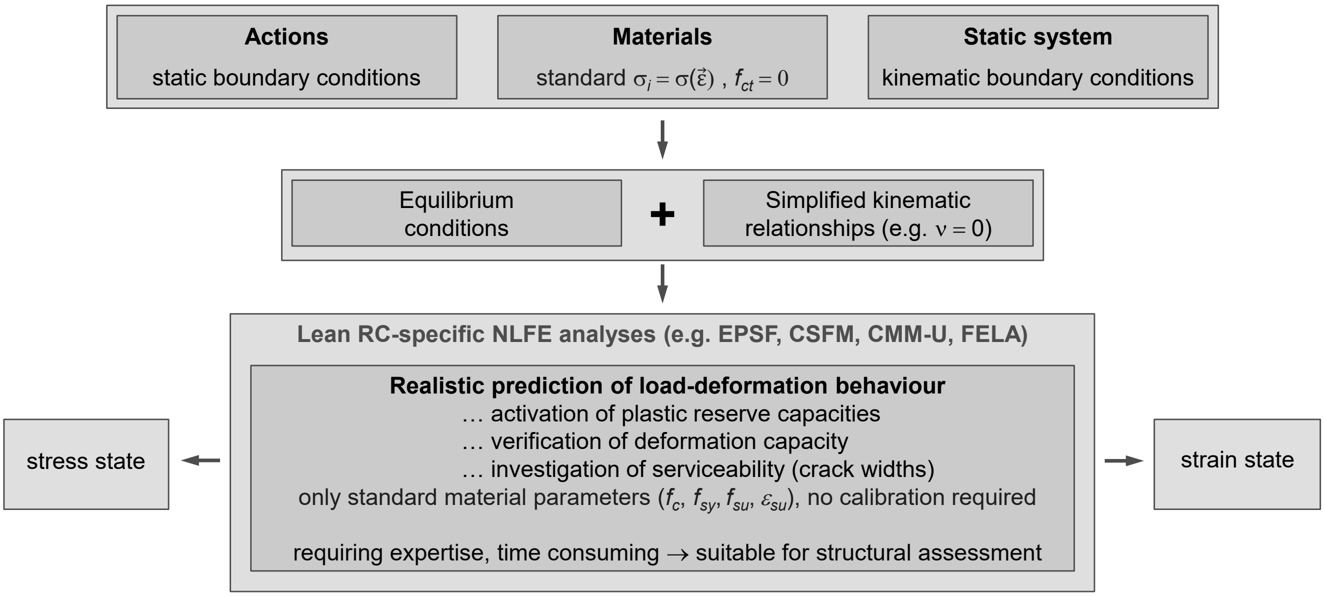




[ATENA NLFE software (Cervenka Consulting)]

Example of a general NLFA analysis. Input masks show some of the non-standard parameters (fracture energy, …) and highlight the numerical complexity (4 convergence parameters in the “simple” method).

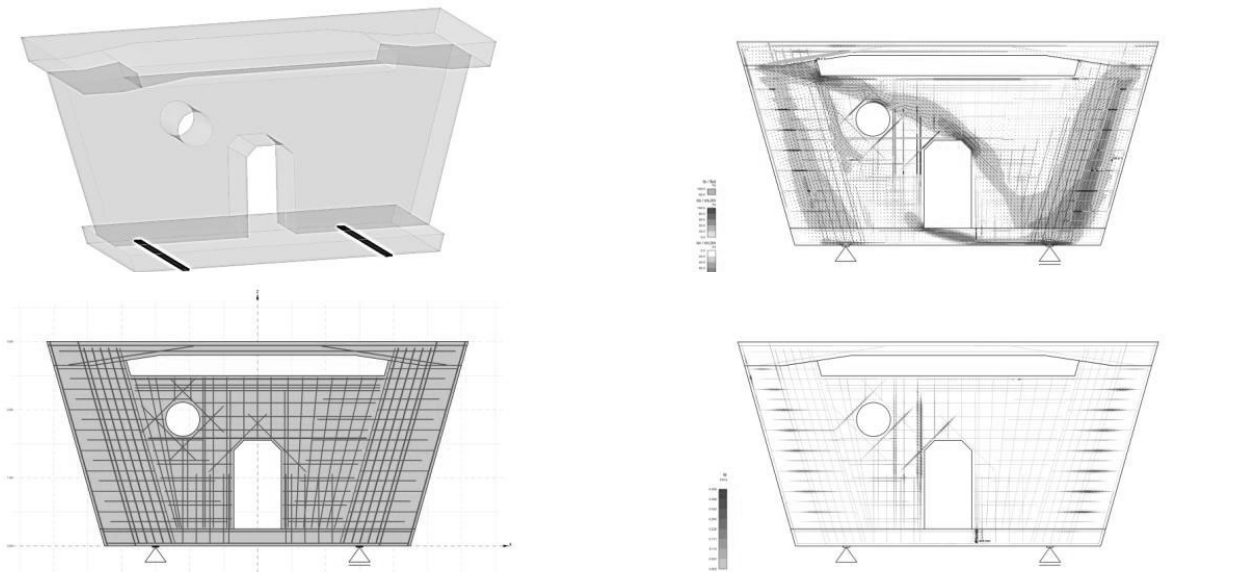
Structural analysis and design – Lean RC-specific NLFE analyses



Lean (i.e., as simple as possible, but sufficient) **RC-specific NLFE analyses** neglect the tensile concrete strength except for tension stiffening and use only standard material parameters without need for calibration.

While still requiring specific knowledge and being time-consuming, they are suitable for assessing existing structures. Furthermore, by neglecting the tensile strength of concrete, they maintain the link to limit analysis methods and are inherently compatible with modern design codes.

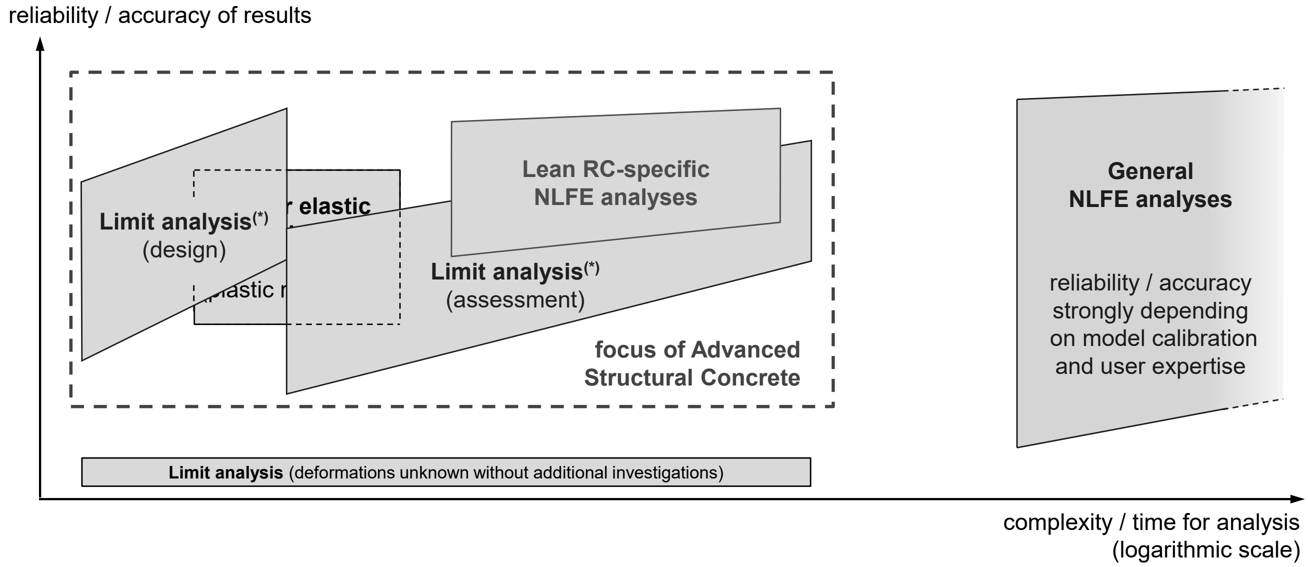
Structural analysis and design – Lean RC-specific NLFE analyses



CSFM (Compatible Stress Field Method, ETHZ / IdeaStatica Detail)
[Davide Bianchi, dsp Ingenieure + Planer AG, 2025]

Example of a Lean RC-specific NLFE analysis (support diaphragm). Strengthening of the diaphragm could be avoided based on the refined investigation with CSFM.

Methods of analysis and design (schematic)



(*) deformations may be obtained by complementing lower-bound solutions with pertinent member stiffnesses, e.g. based on the Tension Chord Model

The slide **schematically** compares the different analysis methods shown.

The slope of the limit analysis methods indicates that accuracy increases with refined models, e.g. stress fields instead of strut-and-tie models, but so does time for analysis.

Fundamentals of limit analysis methods

Theory of plasticity – Limit analysis

Lower-bound (static) theorem

Every loading for which it is possible to specify a statically admissible stress state that does not infringe the yield condition is not greater than the limit (ultimate) load.

(a statically admissible stress state must satisfy equilibrium and the static boundary conditions)

Upper-bound (kinematic) theorem

Every loading that results from equating the work of external forces for a kinematically admissible deformation state with the associated dissipation work is not less than the limit (ultimate) load.

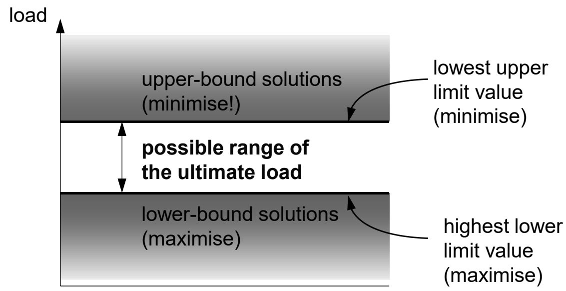
(kinematically admissible: kinematic relationships and kinematic boundary conditions are fulfilled)

Compatibility theorem

A load for which a statically admissible stress state that does not infringe the yield condition and a compatible kinematically admissible state of deformation can be specified is a limit (ultimate) load

The force and deformation states linked by this theorem constitute a *complete solution* to the respective problem.

NB. If upper- and lower-bound solution coincide, the ultimate load has been found – no need to verify compatibility, see figure.



The lower bound theorem of limit analysis guarantees a safe design, provided the applicability conditions are fulfilled (i.e. sufficient deformation capacity). Most design methods in structural concrete are therefore based on the lower bound theorem of the theory of plasticity. The theory of plasticity ensures the safe application of several design methods, for example:

... strut-and-tie models and stress fields
... equilibrium solutions for slabs (e.g. strip method)
... yield conditions for membrane elements and slabs
... etc.

Finding a complete solution is difficult (if not impossible, unless numerical approaches are used), but the advantage of the compatibility theorem is that it does not require performing neither a detailed compatibility check nor a plasticity verification for the mechanism found.

Theory of plasticity – Limit analysis

Prerequisites of the theorems of limit analysis

Strictly, the theorems of limit analysis are valid for perfectly plastic behaviour with:

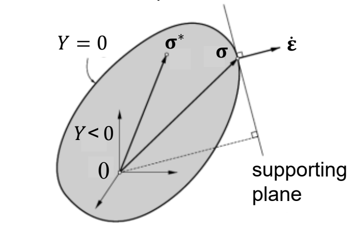
- a (weakly) convex yield surface $Y(\boldsymbol{\sigma}) = 0$ and
- an associated flow rule:
  $\dot{\boldsymbol{\varepsilon}} = \kappa \,\mathrm{grad}\, Y \;(Y = 0\!: \kappa \geq 0; Y < 0\!: \dot{\boldsymbol{\varepsilon}} = 0)$

The latter stipulates that (i) plastic strain increments $\dot{\boldsymbol{\varepsilon}}$ occur only for stress states on the yield surface (rigid-perfectly plastic behaviour) and (ii) these strain increments $\dot{\boldsymbol{\varepsilon}}$ are orthogonal to the yield surface *Y*.

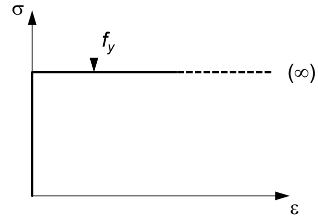
The two conditions of convexity and orthogonality are equivalent to the principle of maximum dissipation:

→ if the yield surface *Y* is (weakly) convex, any plastic strain increment $\dot{\varepsilon}$ generates its maximum dissipation (scalar product $\mathrm{d}D = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}$) for the stress state(s) where it is orthogonal to *Y* (see top left figure)

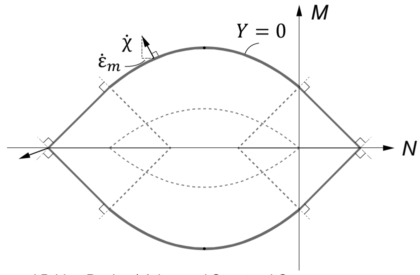
convexity + orthogonality = maximum dissipation:

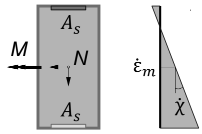


assumed material behaviour: perfectly plastic



example: M-N interaction (generalised stresses {*M*,*N*} and strains $\{\dot{\varepsilon}_m, \dot{\chi}\}$): (see BSc lecture Stahlbeton I)





$$Y_c = \pm M_{yc} + N_c\left(\frac{h}{2} + \frac{N_c}{2bf_c}\right) = 0$$

$$\pm\frac{\dot{\varepsilon}_m}{\dot{\chi}} = \frac{h}{2} + \frac{N_c}{bf_c} = \frac{\partial Y_c / \partial N_c}{\partial Y_c / \partial M_{yc}}$$

Additional remarks to the figure:

A generalised reaction $r = \sigma_i$ can take arbitrary values; this corresponds to a projection of the yield surface into any plane σ_i = const. (for example $\sigma_i = 0$).

The principle of maximum dissipation energy requires the yield surface to be convex and the strain increments to be orthogonal to the yield surface (a two-dimensional graphical example could be used to verify that the dissipation work is not maximum for concave areas of the yield condition or non-orthogonal strain increments).

On the other hand, convexity and orthogonality follow when maximum dissipation energy is given.

Theory of plasticity – Limit analysis

Prerequisites of the theorems of limit analysis

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The latter stipulates that (i) plastic strain increments $\dot{\boldsymbol{\varepsilon}}$ occur only for stress states on the yield surface (rigid-perfectly plastic behaviour) and (ii) these strain increments $\dot{\boldsymbol{\varepsilon}}$ are orthogonal to the yield surface Y.

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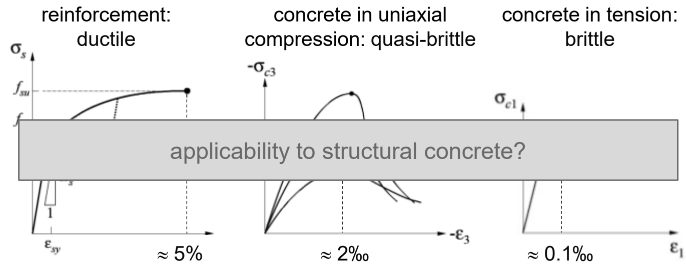
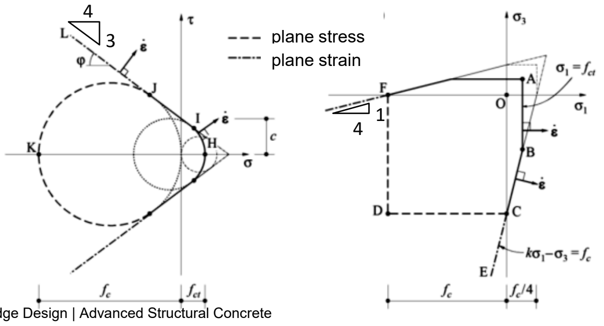


common yield criterion for concrete: Modified Coulomb with tension cut-off (tanφ = 0.75 → $c = f_c/4$, φ ≈ 37°, usually $f_{ct} = 0$)



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On the other hand, convexity and orthogonality follow when maximum dissipation energy is given.

Theory of plasticity – Limit analysis

Prerequisites of the theorems of limit analysis

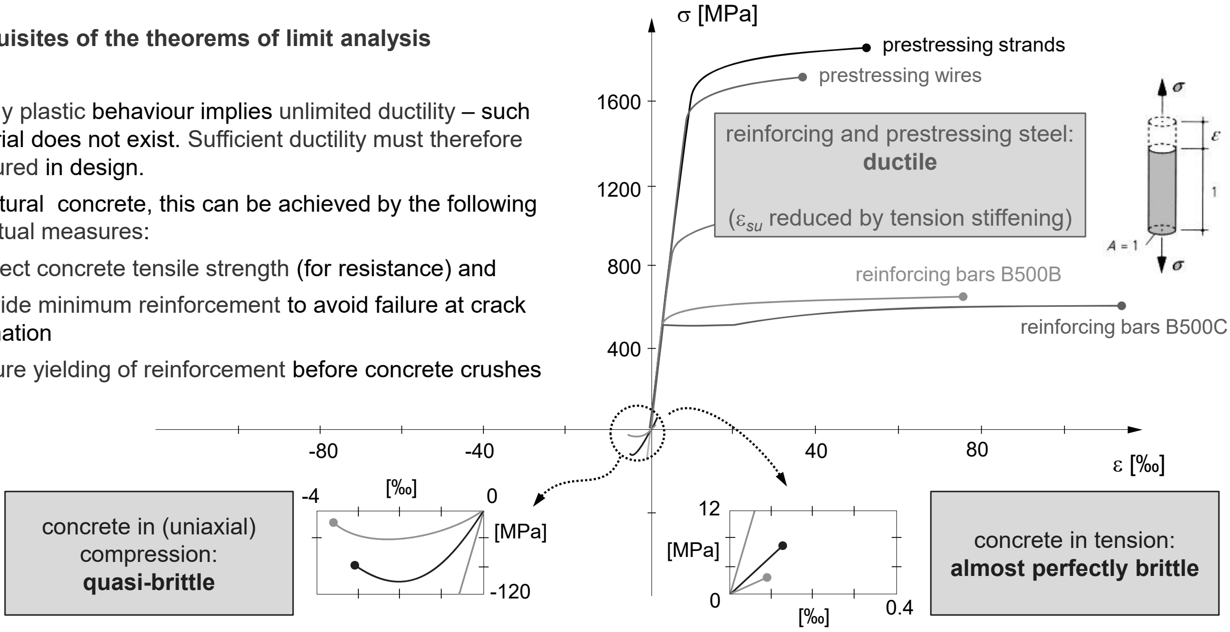
Perfectly plastic behaviour implies unlimited ductility – such a material does not exist. Sufficient ductility must therefore be ensured in design.

In structural concrete, this can be achieved by the following conceptual measures:

→ neglect concrete tensile strength (for resistance) and

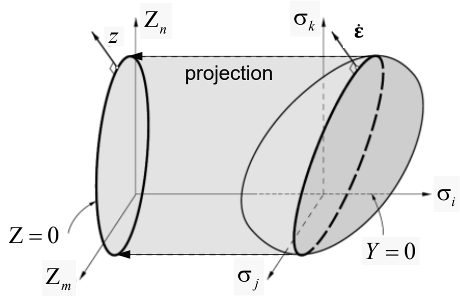
→ provide minimum reinforcement to avoid failure at crack formation

→ ensure yielding of reinforcement before concrete crushes



Theory of plasticity – Limit analysis

Generalised stresses and strains



The theorems of limit analysis equally apply in terms of generalised stresses and strains (deformations).

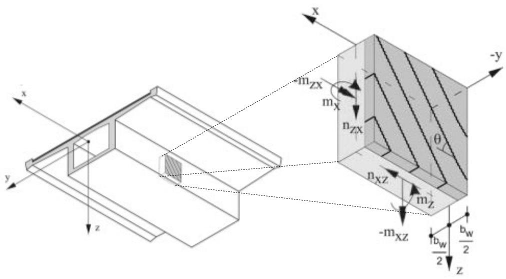
Generalised strains are obtained by introducing kinematic restrictions to the plastic strain increments $\dot{\boldsymbol{\varepsilon}}$. The generalised stress σ_j associated to a generalised strain $\dot{\varepsilon}_j$ results by integrating the stresses $\boldsymbol{\sigma}$ doing work on $\dot{\varepsilon}_j$.

Graphically, introducing kinematic restrictions corresponds to a projection of the yield surface *Y* to a lower-dimensional space *Z*, see top figure:

- projected values = *generalised stresses* {*Z*} *and strains* {*z*}
- stress components σ_i "lost" in the projection = *generalised reactions*, which assume the value maximising the ultimate load in terms of {*Z*}.

Example 1: Bernoulli's hypothesis (*M-N* interaction, see previous slide):

- axial strains $\dot{\varepsilon}_x$ → generalised strains { $\dot{\chi}$, $\dot{\varepsilon}_m$}
  axial stresses σ_x → generalised stresses {*M*, *N*} ("stress resultants")
  shear stresses τ_{zx} → generalised reaction {*V*}



Example 2 (bottom fig.): Shear + transverse bending, see Bridge Design:

- generalised stresses and strains: {n_z(=0), n_{zx}, m_z} and {$\dot{\varepsilon}_z$, $\dot{\gamma}_{zx}$, $\dot{\chi}_z$}
  generalised reactions: {n_x, m_x, m_{zx}, v_{yx}, v_{yz}}

Additional remarks to the figure:

A two-dimensional graphical example can be used to verify that the dissipation work is not maximum for concave areas of the yield condition or non-orthogonal strain increments.

Theory of plasticity – Limit analysis

Main consequences of the theorems of limit analysis

- Residual stresses and restraints have no influence on the ultimate load (as long as the deformations remain small).
  (NB: in elastic solutions and particularly in stability problems, the failure load depends on residual stresses and restraints)
- Adding (subtracting) weightless material cannot decrease (increase) the ultimate load.
- Raising (lowering) the yield limit of the material in any region of a system cannot decrease (increase) its ultimate load.
- The ultimate load determined with a yield surface circumscribing (inscribing) the effective yield surface is an upper (lower) bound to the effective ultimate load.

Application of the theorems of limit analysis

The lower bound theorem of limits analysis is the most used in practice. Typical applications: strut-and-tie models and stress fields for membrane elements, the strip method for slabs.

Many national and international codes are based (in most cases only implicitly, and unknown to many people) on the lower bound theorem.

The upper limit theorem is particularly useful in assessing the structural safety of existing structures. An upper bound of the ultimate load can often be found with considerably less effort than required for the development of a statically admissible stress state that nowhere infringes the yield condition for given dimensions and reinforcement layout).

Pertinent assessment of structural safety

Pertinent assessment of structural safety

Solution:
maximise service life with
minimum interventions
- without excessive / unacceptable risk
- with limited resources

→ pertinent assessment of structural safety is key

Structural safety		**Reality**	
		not ok	ok
Analysis	not ok	ok	resources
	ok	risk of collapse reputation	ok

Pertinent assessment of structural safety

- Analytically often much more demanding than the design of new structures
- High theoretical knowledge and experience required
- Verifications should only be carried out by qualified and experienced structural engineers.
- In practice, the complexity is often underestimated
  (similar analysis than for a new structure or by “reproducing" the structural verifications for construction)